OPTIMAL PORTFOLIOS OF CORPORATE BONDS AND HOLD TO MATURITY STRATEGIES

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ABSTRACT. In this paper we explore optimal bond portfolios that are held to their maturity. We formulate and solve the problem analytically for log utility investor in the case of one risky corporate asset. We compare the behavior of these portfolios to equal weighted portfolios and document out-performance of the optimal portfolios we constructed. In the second part of the paper we utilize simulation based on Vasicek’s Copula approach to derive optimal weights for a corresponding problem involving more than one corporate bond. We discover that these portfolios out-perform naive investment in constant maturity bond indices that have the same maturity horizon. We explain possible application for ALM (Asset Liability Management) strategies.

1. INTRODUCTION

The portfolio optimization problem is one of the central problems in modern finance. The fundamental paper of Markovitz (1952) defined the optimization problem using quadratic utility functions. This paper had a tremendous impact and became a benchmark in portfolio management for practitioners and academicians alike.

The input to optimized weights are expected returns and the covariance matrix of the underlying assets. Thus, to construct optimized portfolios one needs reliable methods to predict returns and estimate covariances between different assets. Returns prediction and the attempt to understand the behavior of assets that grew from Markovitz’s paper occupy a major branch in modern finance. Numerous papers are dedicated to these questions and the literature is too vast to mention. Merton (1980) investigated the confidence intervals for return prediction of stocks. He argued that to estimate returns with any degree of reasonable confidence requires long series of returns. However during this time economic policy and regimes may shift and thus, it becomes a challenging problem.

Uppal, De Miguel, Garlappi (2009) articulated Merton’s claim and demonstrated experimentally that the inherent difficulty to predict returns invalidates the performance of optimized stock portfolios. They computed the performance of optimized portfolios utilizing 14 optimization models that appear in the literature and were

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The topic of this paper originated from discussions the author had with Dr. Elvis Galic and Mr. Klaus Chavanne and later Mr. Daniel Satchkov. Many thanks for their patience and exchange of ideas and discussions. The paper reflects solely authors views and doesn’t reflect view of any institutions that the author is currently affiliated with or was affiliated with in the past.
claimed as improvements to the original Markovitz’s framework. They compared the performance of these model portfolios with a simple \( \frac{1}{N} \) rule they used to allocate weights to different stocks in their portfolio. They discovered that optimization methodologies fail to outperform the simple equal weight allocation in a consistent manner. The authors compared the performance of these portfolios not in sample but out of sample and explained their results by the difficulty in predicting returns for many assets.

Uppal, De Miguel, Garlappi (2009) performed their analysis on portfolio of equities. A natural direction to expand their research is to consider financial assets of different type, like bonds. Thus, we can ask whether a similar phenomena exists for bond portfolios, i.e. whether optimal bond portfolios will fail to out-perform equally weighted portfolios similar to the optimal stock portfolios.

This question was partially answered in a forthcoming paper by Jacobs, Muller and Weber (2013). In this paper the authors expanded the study performed by Uppal, De Miguel, Garlappi (2009) and considered different types of assets in particular bonds and commodities. They compare simple heuristic investment rules like the equal weight strategy and discover that optimal portfolios fail to out perform those portfolios in out of sample analysis. Jacobs, Muller and Weber (2013) conclude that the phenomena observed in equities occurs also in other asset classes. That is, they conclude that the mean variance optimization doesn’t out-perform simple ad-hoc allocation rules like an equal weight allocation out of sample. While the study is comprehensive and includes commodities which is an asset that usually isn’t considered in asset allocation studies, there are some natural questions that these studies raise:

- The holding window of the portfolios is one month. Is it possible that different holding windows will generate different results?
- The study focused on asset types and not sub-sectors of certain asset type. Is it possible that different sectors within an asset type will exhibit a different behavior?
- Are all simple rules for investing created equal or are there investing rules that may be suitable to achieve certain investment goals?

For fixed income securities these questions are interconnected. Unlike equities, fixed coupon bonds have varying maturity horizons. Maturity horizons are the leading factors of bond behavior versus the underlying yield or spread structure. Thus, it may happen that optimal bond portfolios with different maturity horizons may exhibit different behavior versus equally weighted portfolio depending on the optimization time horizon. The indices that Jacobs, Muller and Weber (2013) used in their paper for bonds are constant maturity indices. Bonds that belong to a constant maturity index, and, are about to mature are replaced with new bonds with maturities dictated by the rules of the index, hence, such bond index never matures.

The tacit assumption made by the authors, to treat bonds as constant maturity instruments enables them to consider fixed income securities as part of Markovitz’s framework. Bond optimization questions are reduced to equity-like techniques. To predict bond returns one needs to predict the term structure of the yield curve of the underlying bonds, similar to predicting equity returns. This can be a challenging task particularly for long return horizons. We note that if we assume a term structure for yield curve than we can consider the dynamic version of optimization;
see for example Bielecki and Jang (2007) and Capponi and Fugeroa Lopez (2013). These papers assume underlying dynamics of the yield curve and solve explicitly for the weights of optimized dynamic allocation between corporate and treasury securities. The new feature introduced by these authors in the allocation problem is the presence of jump to default, to consider the risk of the underlying corporate bond defaulting.

As we shall see below our paper differs from the theoretical investigations initiated by Bielecki and Jang (2007). First these authors consider a dynamic version of bonds optimization while we consider 1 period problems of bond optimization. Secondly these papers are theoretical in nature. The authors haven’t calibrated and tested the models in reality. Our main task in this paper will be the practical application of optimization ideas to improve returns investors can expect engaging in a concrete strategy. Our paper is closer in spirit to the line of research initiated by Jacobs, Muller and Weber (2013). Finally the solution produced by Bielecki and Jang assumes a presence of Constant Maturity Bond Indices (CMI) consequently, to implement the model one is forced to produce a projection of the future yield. In our optimization set up we aren’t attempting to project spreads or bond yields but instead we rely on the bond’s current yield to solve the optimization problem. In this paper we consider a portfolio optimization problems, assuming that investors hold positions in portfolios until maturity. This implied that they don’t trade these positions during the holding period. We compare these optimal portfolios with the corresponding equal weighted and corporate index portfolios in the spirit of Jacobs, Muller and Weber (2013). We find out that in the case of hold to maturity strategies optimal portfolios tend to outperform portfolios with simple rules we consider and the out-performance is out of sample. Thus, the additional feature we introduced (holding positions till their maturity) implies different results from the results in the papers we mentioned above. This has applications to liability management, because most of the optimization is done using the constant maturity approach, while the process we outline may present an interesting alternative.

Let us assume that the bonds are risk free bonds with a certain fixed term maturity. In this case we don’t need to estimate any parameters because the future cash flows of the bond and its final payment are completely determined by the term and conditions that define the bond at the time of the purchase. Thus, for bonds with no risk that are held to maturity no optimization is needed. In particular we don’t need to estimate any future interest rates.

What are risks for such a strategy? The only two risks we encounter in the case of HTM (Hold to Maturity) strategies are:

1. Bond issuer have not met its obligations and defaults
2. The bond principal was refinanced prior to maturity

An example for bonds with first type of risk are bullet bonds whose cash flows are entirely predictable and the only source of risk is the default risk. The typical example for bonds with irregular payments are mortgage bonds. In this paper we focus on the first type of risks and don’t consider bonds with irregular cash flows like mortgages. We analyze the performance of optimized portfolios that mainly consist of corporate bonds. We assume a fixed holding time to maturity of one year. We replace the usual mean variance framework with power utility optimization because the one period problem we analyze is non-normal in nature. We compare different

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2 We plan to investigate the behavior of mortgage bonds in a later paper.
rules of portfolio allocation and their performance with treasury benchmark and their performance among themselves, in particular, with simple rule allocations for example, $\frac{1}{N}$, or an investment in the constant maturity corporate index. We find out that holding to maturity matters and optimal portfolios seem to out-perform portfolios with naive allocation rules.

While the questions addressed in this paper are relevant to portfolios with hold to maturity strategies (HTM), most of the indices and bond portfolios that are used in practice are constant maturity indices (CMI). In the paper we show an example how we can apply our analysis to constant maturity bond indices and implement a simple dynamic strategy to outperform certain constant maturity treasury (CMT) indices with a high probability. These strategies take advantage of the way constant maturity indices are constructed and are examples of wide range of tactical strategies used by traders to out-perform constant maturity indices that are known as "front running".

The structure of the paper is as follows. First, we explain the general construction of indices and elucidate the differences between Hold to Maturity and Constant Maturity indices. First, we consider a HTM optimization problem where our bonds have spread over the risk free rate but have virtually zero risk of default. We compare this to Markovitz’s optimization technique and we will see major differences in behavior of the two. We explain how to use the HTM results to construct a dynamic strategy for the index with constant maturity bonds that is guaranteed to outperform the CMT treasury index and thus, generate a non trivial alpha. Albeit with few basis points. As far as the author knows these kind of strategies didn’t appear in the literature previously. The alpha generated is model free. Next we turn our attention to the major object of our study, that is, one period optimal portfolios of corporate bonds with varying probability of default. We measure default risk based on their rating. We formulate a one period optimization problem for a log investor and solve it analytically. We apply this solution to construct optimal portfolios for different rating categories and compare these portfolios with the corresponding equal weighted portfolios. We carry out a similar procedure based on the Markovitz’s framework for constant maturity index portfolios. We compare the weights and show that HTM optimal portfolios have much more realistic weights. Next we investigate the sensitivity of the optimized weights to the risk aversion coefficient and carry out an optimization for power utility functions with a risk aversion coefficient different from 1. One interesting finding supports the conclusion of Uppal, De Miguel, Garlappi (2009) and shows that on a long time horizon an equal weighted equity portfolio will be optimal for investors with realistic risk aversion.

Finally we turn our attention to the multi-asset case. Here we utilize Vasicek’s model of mutual default correlations to formulate an optimization problem for more than one corporate bond. For small number of bonds we solve this problem numerically and compare the performance for equal weighted portfolio invested in those bonds and a CMI that consists of corporate bonds with high rating. Our finding seems to confirm that even in this case optimal portfolios we constructed out-perform the equal weighted and index portfolios out of sample. We conclude by discussing our findings and outlining how to expand and broaden the research carries in this paper.
2. HTM versus CMT indices

In this section we explain how various bond indices are constructed. We use this to produce various strategies that "front run" the index and generate non-trivial alpha in a model free way.
In general, fixed income indices are constructed based on the following criteria:

1. Liquidity requirements - usually given through the minimal issue size that can enter the index
2. Maturity requirements - these include bonds that satisfy certain maturity criteria. (If a bond in question has embedded options, for example embedded put or call dates these are ignored and only its actual maturity is considered.)
3. Sector - Fixed income bonds will adhere to certain well defined sector breakdown criteria. These are based on the industry classification defined and maintained by the index provider.
4. Rating Requirements - These include bonds that adhere to certain rating requirements

The four categories above define the indices almost uniquely. The administrator of the index publishes index rules for the use of asset managers and investors alike. We provide typical examples of such indices below:

2.1. **US Aggregate Fixed Income Index.** The most general index which is the benchmark for many investors is the US Aggregate index maintained by Barclays. The rules of the index are:

1. Liquidity - The index includes all issue sizes of publicly traded bonds in the U.S. greater equals of 1 Billion Dollars
2. Maturity - Any bond more that has more than one year to its maturity
3. Sector - all sectors of taxable bonds traded and publicly registered in the U.S.
4. Rating - Investment Grade bonds (defined as BBB and above rating by the index provider)

The re-balancing of the index is done once a month where new bonds will enter and exit the index based on their maturities and other criteria outlined above.

2.2. **US Treasury Index.** The rules that define the by the index provider (Bank Of America) are:

1. Liquidity - The index includes all issue sizes of publicly traded bonds of US government which is greater or equals 250 million
2. Maturity - Any treasury bond that has at least one day until maturity
4. Rating - Irrelevant because the US government has a well defined rating by the rating agencies.

The US treasury index is frequently divided into maturity buckets to address different investment horizons for various investors. In our first example we will concentrate on the short term indices i.e. the treasury index that included treasury bonds up to one year of maturity and treasury indices for that have bond maturing in the range of 1 to 3 years.

3. **Portfolio Allocation with Agency Bond and Indices**

To show the difference between optimized portfolio that are created based on HMT strategies versus optimized portfolio that are created by CMI strategies, consider an optimal allocation for bonds issued by U.S. Agencies. These institutions are implicitly supported by the US government. Examples of such agencies include:
These entities provide fund different social or infrastructure activities sponsored by the US government. The bonds issued by the Agencies are guaranteed implicitly or explicitly by the US government. However the debt of the Agencies is cheaper than a parallel US government debt with similar maturities.

Consider an index comprised of short term Agency debt. The spread of the index closely approximates the difference between the average yield of bonds comprising the index and the corresponding yield of treasury indices with similar maturities. Below we provide a historical graph of short term spreads of the index versus treasury:

**Figure 1.** This graph describes the spread of short Agency spread 03/30/2004-03/31/2014
The table below details the main statistical properties of the spread time series:

**Table 1. Main statistical properties of the Spread Index**

<table>
<thead>
<tr>
<th></th>
<th>OAS</th>
</tr>
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<tbody>
<tr>
<td>Average Spread</td>
<td>0.30</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>0.91</td>
</tr>
<tr>
<td>Max Spread</td>
<td>1.66</td>
</tr>
<tr>
<td>Min Spread</td>
<td>0.081</td>
</tr>
</tbody>
</table>

Consider an allocation problem where we have a treasury bond maturing a year from now and a corresponding Agency bond. We are interested in maximizing our investment through the horizon time of the bond maturity. If we assume an allocation problem in which only positive weights are allowed than an immediate solution will be to simply put our entire investment into the Agency bond. This setup carries virtually no risk. (remember that Agency bonds are supported by the US government.) Conclude that a hold to maturity strategy guarantees out performance of any portfolio which will be a mix of agency and treasury bonds maturing at the same horizon.

The table below summarizes the statistical properties of the strategy. The out of sample statistical properties coincide with the statistical properties of the series because nothing has to be estimated. We compute the following statistics:

- Sharpe ratio
- Power utility using for the utility function: \( \frac{w^3}{3} \)

In our calculation we make the following assumptions:

- Investment horizon of 1 year
- Initial investment in the beginning of period of 1 dollar
- Continuous returns to simplify formulas

We assume an investment horizon of one year, and, continuous return when we calculate final wealth and it’s utility.

**Table 2. The table describes the statistical properties of the strategy. The strategy is always positive.**

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
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</thead>
<tbody>
<tr>
<td>Average Spread</td>
<td>0.30%</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>0.26%</td>
</tr>
<tr>
<td>Max OutPerformnace</td>
<td>1.66%</td>
</tr>
<tr>
<td>Min Outperformance</td>
<td>0.08%</td>
</tr>
<tr>
<td>Sharpe Ratio</td>
<td>1.16</td>
</tr>
<tr>
<td>Average Utility</td>
<td>0.50</td>
</tr>
</tbody>
</table>

The strategy outlined above is most profitable when the spread of corresponding agency bonds is most elevated. This usually happens through an economic crisis. Note that the strategy doesn’t involve explicit prediction of the future Agency spread nor the calculation of any statistical properties of returns. Any strategy
that allocates less than the full weight into the short term agency index will clearly under perform.

While we observe this opportunity in the market it is interesting to investigate why such an opportunity exists at all. If Agency bonds supported implicitly by the US government why aren’t these strategies used by market practitioners on a massive scale? This was already noted by Longstaff (2002) who documented a similar phenomena with the RefCorp corporation that was formed to retire MBS portfolios after the Saving and Loans scandal. Longstaff explained spread between Agency bonds and treasury bonds is because of liquidity. The issuance of the treasuries and it’s market infrastructure is more developed than the corresponding market for short Agencies. Thus, traders will forgo the extra performance they can obtain from holding Agency bonds for the security of being able to liquidate the positions if they need to. We remind the reader that out-performance we calculated is only valid for HTM type strategies. For strategies in which the investor needs to sell his holdings prior to maturity of the Agency bond we may lose money versus the corresponding treasury bonds. Hence for investors that don’t utilize HTM strategies to manage their fixed income book, (currently this is the vast majority) this opportunity won’t exist at all.

4. Short Term Agency and Constant Maturity Index

A natural question we can ask whether we can modify the strategy above to the case of constant maturity indices. We show that there is a natural extension if we make an assumption about the re-balancing strategy of the treasury index. Recall that an index has to be re-balanced to accommodate maturing bonds or to absorb new debt offering that are complying with the rule of the index. Assume ideally that the government issues debt once a year. Then the index is re-balanced once a year because bonds mature at once. In we employ the following strategy that guarantees out-performance:

- At the initial re-balancing date buy the bonds (or their subset) of bonds that belong to the Short Agency Index
- At the moment of index re-balancing buy more bonds (or their subset) that belong to the Short Agency Index.
- Continue this dynamic strategy indefinitely.

In this type of strategy the out-performance would be guaranteed only by the end of the year. The price of the bond portfolio can fluctuate during the year but will eventually converge to it’s face value. Consider an investor who invests into the short term Agency index as a constant maturity index. He buys the index at a certain date and measures his performance based on the price (and income) movements of the index. This investor will not be aware of any re-balancing occurring with the securities comprising the index. The table below compares the strategy we suggest (rolling HTM) with the strategy of investing directly into the short term Agency constant maturity index:
Table 3. Table Summarizing the performance of the return above treasury of HTM strategy with the corresponding strategy of investing in the CMT index

<table>
<thead>
<tr>
<th></th>
<th>OAS</th>
<th>Excess Return (12 month Rolling)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average Spread</td>
<td>0.30</td>
<td>0.42</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>0.26</td>
<td>0.66</td>
</tr>
<tr>
<td>Max Spread</td>
<td>1.66</td>
<td>3.08</td>
</tr>
<tr>
<td>Min Spread</td>
<td>0.08</td>
<td>-1.29</td>
</tr>
<tr>
<td>Sharpe Ratio</td>
<td>1.16</td>
<td>0.64</td>
</tr>
<tr>
<td>Average Utility</td>
<td>0.50</td>
<td>0.50</td>
</tr>
</tbody>
</table>

Comparing the average utilities we observe that investors with risk aversion of 3 are indifferent whether to invest in HTM or CMT index. The average excess return of constant maturity index is higher than average spread of the HTM strategy but Sharpe ratios of the HTM strategy are higher than investing in constant maturity index.

We assumed a simplified model of the index re-balancing. In reality the indices are re-balanced on a monthly basis. But the strategy we described for annual re-balancing of the treasury (and Agency index) can be applied to monthly re-balances of any treasury index. While we don’t have a good historical data on the breakdown of historical Agency spread from inception to one year horizon we can assume a linear structure and on average monthly re-balancing strategy should yield approximately 15 basis points of out performance. We remark that this strategy always yields positive out-performance of the treasury index.

The technique we described is popular among portfolio managers that have a certain benchmark as a target they need to out-perform. The technique is called index front running and is based on the fact that the index re-balanced on a monthly basis and the new securities that are included in the index are known. In this case the trader buys the relevant security before the re-balancing date assuming its price rises once it enters the index. This strategy is closely related to the strategy we used.

5. Optimal Portfolios with Corporate Default Risk

5.1. Model Set Up. In the previous section we produced an example in which we outperform the treasury index using bonds that have no bankruptcy risk and their yield is above the treasury yield. In this section we consider how to optimally allocate to bonds that can default. We assume that these are corporate bullet bonds and we hold these bonds to maturity. The risk free asset consists of a treasury index that matures at the same time or close to the maturity of the corporate bond. The maturity is always one year from the optimization period. Defaults aren’t distributed normally, hence we consider a power utility optimization problem using the log. Equivalently we assume that risk aversion coefficient, $\gamma = 1$.

Assume we invest $w_1$ in the risky bond and $1 - w_1$ in the risk-less bond. The price of the risky bond today is $P_1$, $y_1, y_2$ are the yields we earn on each of the bonds and $R$ is the recovery rate of the corporate bond. In the case of default the
value of the portfolio is:

\[ R \frac{w_1}{P_1} + (1 - w_1) \times \exp(y_2T) \]

and in the case of non default we have:

\[ w_1 \exp(y_1T) + (1 - w_1) \times \exp(y_2T) \]

Thus, we are seeking to optimize the following:

\[ \beta_1 \log \left( R \frac{w_1}{P_1} + (1 - w_1) \times \exp(y_2T) \right) + (1 - \beta_1) \times \log \left( \frac{w_1}{\exp(y_1T)} + (1 - w_1) \times \exp(y_2T) \right) \]

Let: \( c_1 = \frac{R}{P_1}, c_2 = \exp(y_2T), c_3 = \exp(y_1T), c_4 = \exp(y_2T) \). We need to optimize:

\[ u(w_1) = \beta_1 \log (c_1 R w_1 + c_2 (1 - w_1)) + (1 - \beta_1) \times \log (c_3 w_1 + c_4 (1 - w_1)) \]

To solve this optimization problem we take the derivative with respect to \( w_1 \) and we get:

\[ \frac{\beta_1 \times (c_1 - c_2)}{c_1 w_1 + c_2 (1 - w_1)} + \frac{(1 - \beta_1)(c_3 - c_4)}{c_3 w_1 + c_4 (1 - w_1)} \]

Clearing the denominator we obtain the following equation:

\[ \beta_1 w_1 \times (c_1 - c_2)(c_3 - c_4) + \beta_1 (c_1 - c_2) c_4 + (1 - \beta_1) w_1 (c_3 - c_4)(c_1 - c_2) + (1 - \beta_1)(c_3 - c_4) c_2 = 0 \]

We can simplify the last expression as:

\[ \beta_1 (c_1 - c_2) c_4 + \beta_1 w_1 (c_3 - c_4)(c_1 - c_2) + (1 - \beta_1)(c_3 - c_4) c_2 = 0 \]

Hence we get that:

\[ \beta_1 w_1 (c_3 - c_4)(c_1 - c_2) = -\beta_1 (c_1 - c_2) c_4 - (1 - \beta_1)(c_3 - c_4) c_2 \]

Divide both sides by \((c_3 - c_4)(c_1 - c_2)\) to obtain:

\[ w_1 = -\frac{\beta_1 c_4}{c_3 - c_4} - \frac{(1 - \beta_1) c_2}{c_1 - c_2} \]

but \(-(c_1 - c_2) = c_2 - c_1\) and \(-(c_3 - c_4) = c_4 - c_3\). Hence the optimal weight exists and equals to:

\[ w_1 = \frac{\beta_1 c_4}{c_4 - c_3} + \frac{(1 - \beta_1) c_2}{c_2 - c_1} \]

Given \( w_1 \) that we can find we can form an optimal portfolio for HTM bond strategy for a certain time horizon and compare the performance of optimal log utility portfolios with the performance of the simple allocation strategy of equal weighting. We explain our method of portfolio performance in the next sections and present the main results.

5.2. Model Calibration Methodology. To calibrate and estimate different default probabilities of bonds we chose to rely on rating agencies. The role of rating agencies is to rate publicly traded bonds in the U.S. The risk of default is quantified by a letter based rating system varying from AAA (the lowest probability of default) to C the highest probability of default. In our estimation of defaults we bucketed bonds with similar quality ratings. Thus, we had the following rating buckets:

(1) AA-AAA
(2) A-BBB
(3) BB
(4) C-CCC

The bucketing was dictated by index availability that contains historical data we needed to estimate the quantities that we used to calculate the weights in our optimized portfolio.

For each bucket we utilized the monthly index series of short term bonds that are in the index and have ratings in the rating bucket we defined above. We used the following index data to calibrate our model:

1. OAS - Option Adjusted Spread - the difference between the average yield of the corporate bonds in the bucket and the yield of the corresponding treasury index. (This is calculated by the index provider)
2. Average risk-less yield - represented by the average yield of the treasury index with maturity bands from 1 year to 3 years
3. Average Price Index - This is the average price of the corporate bonds in the index buckets with maturity 1-3 we defined as approximation for the short term bonds
4. Weighted average maturity - as a control variable to verify that maturities of corresponding indices match
5. Effective Duration and Spread duration - to quantify the risk exposure to interest and spread rate.

Equation 10 is the formula to the risky asset $w_1$. The inputs to the equation are based on the monthly time series of the historical indices data. More precisely we have:

1. $P_1$ - the level of the price index of the corresponding risky bond for a given month
2. $P_2$ - the level of the treasury index with the same maturity buckets as of the risky bond
3. $y_2$ - Treasury yield of the index at a given month
4. $y_1 = y_2 + OAS$ - Proxy to the yield of the risky bond.

Our historical corporate index data starts at 1996, and therefore, 1996-2013 is our optimization period. Estimating default time series is a challenging problem because these are rare events, hence, we assumed that the probability of default is constant. For each rating cohort we calculated the average default rate from 1920-1996 and used it for the entire optimization period 1996-2013. As the the price series are monthly and the default rate series are annual, we assumed a constant default rate for the 12 monthly data points that belong to a certain year.

We assumed a recovery rate of 40% for investment grade and 30% recovery for high yield bonds. The source of these assumptions are the standard recovery assumptions in the credit default swap market. We didn’t find any historical data for the price of corporate bonds at default as this is the true recovery rate. We note that recovery rates aren’t systematic but rather specific and may vary significantly between different bankruptcy events.

5.3. Portfolio Construction. To construct the portfolio we use equation (10) to calculate the risky weight $w_1$. We use monthly index data as the inputs to the model. We obtain the quantities that are present in equation (10) from monthly indices as we explained above. More precisely we form portfolios on a monthly
basis using weights $w_1, 1 - w_1$. That is, we assume we have one currency unit to invest, and, we invest $w_1$ into the risky asset and $1 - w_1$ into the non risky asset, the corresponding treasury index.

Once we invest in the portfolio we don’t perform any re-balancing action on it until they mature. We calculate their synthetic performance and their utility based on the actual number of defaults that occurred during this year (we explain in the next subsection we perform this adjustment.). Similarly we form an equally weighted portfolio (that is we invest $\frac{1}{2}$ of the currency unit in corporate bonds and in the risk-less index) and calculate its performance and utility.

As the construction of our portfolio is on a rolling month basis (that is, each month we form a portfolio and liquidate precisely one year later not performing any trades in the middle.) our results are automatically out of sample. A careful examination of the formulas for $w_1$ reveals that these depend only on the data available to the investor in the present and any additional information that is available only in the future isn’t used. We assume our default projection to be the average default in the period 1920-1990. Hence an investor pursuing our optimal strategy doesn’t use any future information in sample that wasn’t available to him at inception time (in 1996.) We obtain out of sample monthly series of performance of log utility portfolios. We calculate the average monthly out-performance and the utility difference and summarize our results.

5.4. Performance Calculation. We explain how we adjust the performance of the calculated portfolios to the actual default numbers we observe in the market and the logic behind our adjustment. Our underlying portfolio construction relies on numbers provided by a synthetic index. The index constituents are bonds that are issued by various issuers. When we construct any portfolio our method is to randomly select a name which is part of the data pool of the issuers on which the rating is constructed. Now assume that we know that the actual default rate was $\alpha$. This means that out of the synthetic bond we selected randomly we have a probability of $\alpha$ to default. i.e. if the yield of the risky bond is $y_1$ with the price of $P_1$, and, the default adjusted amount of the risky bond is going to be:

\[(1 - \alpha) \times w_1 \times \exp(y_1T) + \alpha \times R \times \frac{w_1}{P_1}\]

for the risk-less bond we are assured to get the full amount that is the full value of our investment default adjusted:

\[(1 - \alpha) \times w_1 \times \exp(y_1T) + \alpha \times R \times \frac{w_1}{P_1} + (1 - w_1) \times \exp(y_2T)\]

The last formula holds in any situation for arbitrary weights. We used it to calculate the final wealth for the weights with optimized and equally weighted portfolios.

5.5. Results. We performed the following procedure to compare the performance of optimal portfolios to the equal weighted portfolios:

1. Estimate the parameters out of sample and construct optimal portfolios.
2. Optimal portfolios consist of treasury index and arbitrary synthetic bond with the spread represented by the index.
3. Assuming that the bond is held to maturity and has a certain yield we can calculate its performance and hence its utility in a straightforward manner.
(4) We subtract from this performance the percentage of the names that defaulted in this year according to the data provided by Moody’s investor services.

(5) We calculate the final utility and performance and compare it with the utility of portfolio that was created equally weighted between treasury and the corporate index.

We repeat this procedure on a monthly basis and obtain time series of log utility functions and performances of this portfolio. We calculate statistics of this utilities for each rating bucket defined above and tabulate the results.

In our model we need to predict only the default probability of each rating bucket. We do it by finding a simple average of the default series obtained from Moody’s annual corporate default study. We used a constant default average for each time series. Once we have the default series we can form the log utility portfolio and compare it with a simple strategy of equal weighted allocation between the benchmark index (the corporate index and the treasury) and the log utility out of sample portfolios that we created using the formula above. The table below presents the assumptions for annual default rate and recovery rates for the rating buckets we investigate:

<table>
<thead>
<tr>
<th>Rating</th>
<th>Recovery Rate</th>
<th>Default Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA-AA</td>
<td>40.00%</td>
<td>0.09%</td>
</tr>
<tr>
<td>BBB-A</td>
<td>40.00%</td>
<td>0.18%</td>
</tr>
<tr>
<td>HYBB</td>
<td>30.00%</td>
<td>1.37%</td>
</tr>
<tr>
<td>CCC-C</td>
<td>10.00%</td>
<td>20%</td>
</tr>
</tbody>
</table>

Based on the assumptions above we form log utility portfolios and compare the performance of these portfolios in each corporate rating bucket with a portfolio that is formed equally weighting the corporate index and the risk-less treasury index. We remind the reader that our benchmark is the treasury index that is held to maturity. Our log utility portfolios are by construction out of sample portfolios with 1 year holding period on a monthly period 1996-2013.

The table below summarizes the performance of portfolios optimized for the log utility problem with HTM strategy versus one year versus the equally weight portfolio formed at the same period. Below we describe the overall synthetic series we created for the out performance (or under performance) of the optimized portfolios versus the equal weight portfolio for different rating cohorts and it’s weights:
Table 5. The table describes the difference between the performance of log utility portfolio versus the equally weighted portfolio for a monthly sample of 20 years. The kurtosis is calculated on the out-performance series of the optimal versus equal weighted portfolios.

<table>
<thead>
<tr>
<th>Rating</th>
<th>Out-performance</th>
<th>Standard Deviation</th>
<th>Min</th>
<th>Maximum</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA-AA</td>
<td>0.63%</td>
<td>0.76%</td>
<td>0.13%</td>
<td>4.83%</td>
<td>14.78</td>
</tr>
<tr>
<td>BBB-A</td>
<td>0.91%</td>
<td>1.18%</td>
<td>-0.40%</td>
<td>7.39%</td>
<td>14.96</td>
</tr>
<tr>
<td>HYBB</td>
<td>2.28%</td>
<td>3.24%</td>
<td>-0.58%</td>
<td>22.29%</td>
<td>15.21</td>
</tr>
<tr>
<td>CCC-C</td>
<td>5.89%</td>
<td>8%</td>
<td>-6%</td>
<td>45%</td>
<td>7.67</td>
</tr>
</tbody>
</table>

Figure 2. This graph describes the difference between the performance of the optimized and the equal portfolios. 1996-2013

Figure 3. This graph describes the optimized historical weights for the different rating cohorts 1996-2013

The surprising feature of the weights given in the last table is the negative weights attributed to C-CCC. While we can rely on the formula for optimal weights in our model we can explain the result in an alternative way. If the default rate probability is higher than the expected income or return we can earn from the bond we will short this instrument because the yield doesn’t provide ample compensation for the possible principle loss we need to absorb. Recall we aren’t trying to predict the
default rate but we assume it to be the long term historical default rate as observed by the rating agencies. This implies that investors in distressed corporate bonds need to have a view fundamentally different from the average default probability we observe historically, otherwise this asset class wouldn’t exist because it will be permanently shorted. Indeed the distressed market is the main market for private equity and hedge fund investors that attempt to sort the winners from the losers in this market.

The second feature is the relatively high kurtosis that our strategy has. The high kurtosis implies that except the C-CCC buckets that is fundamentally different our investment strategy has essentially the same kurtosis for the rest of the buckets. This provides an interesting supporting evidence to opinions prevalent among professional traders that trading bonds in lower capital structure resembles more equity like trading because they are less sensitive to interest rate and yield movements, due to the high risk of default. In the next section we analyze certainty equivalents of the different strategies. We discover that certainty equivalent of our optimized strategies are better than certainty equivalents of equal weighted portfolios. We remind the reader that according to utility theory certainty equivalence measure provide a better approximation of investor’s risk appetite than distribution first moments.

6. Certainty Equivalent Portfolios Comparison

We showed that log utility optimized portfolios outperform the equal weighted and benchmark portfolios. Let us compare the portfolios based on their riskiness. As we use log utility portfolios we compare them based on the rolling certainty equivalent. Our methodology is similar to Uppal, De Miguel, Garlappi (2009). That is we create a window of 24 months and use it to calculate a rolling average for the logarithmic function of the final wealth based on the optimized portfolio and the equal weighted portfolio. We use this rolling average to calculate the 24 month certainty equivalent for the optimized and equal weight portfolio. We documented the average of these rolling average similar to the paper mentioned above. We performed this comparison for each of the rating buckets and we display the results above. The graph below represents the difference between certainty equivalent of the rolling average for the optimized and the equal weight portfolio for the historical sample of 18 years (1996-2013):

The statistical properties of certainty equivalents are given in the table below.
This graph describes the difference between the Certainty Equivalents of optimized versus the benchmark (equally weighted portfolios) from 1996-2013.

Table 7. This table describes the statistical properties of Rolling Average of Certainty Equivalent of optimized and equal weighted portfolios.

<table>
<thead>
<tr>
<th></th>
<th>Optimized</th>
<th>Equal Weight</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>AA-AAA</td>
<td>1.085</td>
<td>1.082</td>
<td>0.28%</td>
</tr>
<tr>
<td>BBB-A</td>
<td>1.051</td>
<td>1.041</td>
<td>0.95%</td>
</tr>
<tr>
<td>HYBB</td>
<td>1.074</td>
<td>1.05</td>
<td>2.4%</td>
</tr>
</tbody>
</table>

The table and the graph show that the out-performance of the optimized log utility portfolio is consistent on average for different historical time periods. Furthermore the out-performance rises as the corporate bonds become more prone to default and hence more risky.

6.1. Certainty Equivalents for other power utility functions. It is well known that log utility portfolios correspond to investors whose coefficient of risk aversion is 1. We investigated what happens when the risk aversion isn’t 1 but higher. log utility becomes power utility function. More specifically consider a power utility function of the form: $W^{1-\rho}$. We can set up the optimization problem for these type of functions exactly as we’ve done above. The corresponding equation for the derivative will be:

$$\beta_1(c_1-c_4) (w_1 c_1 + (1-w_1)c_4) - \rho + (1-\beta_1) (c_2-c_3) \times (w_1 \times c_2 + (1-w_1) \times c_3) = 0$$

While this equation can’t be solved analytically for a generic $\rho$ we can solve it numerically and obtain exactly the same results as in the case of $\rho = 1$. For example the table below describes the statistical properties for $\rho = 2$:

Notice that the average out-performance for the rating buckets is less than the out-performance of the optimal portfolios corresponding to the log investor ($\rho = 1$.) This agrees with the theory, since people with higher risk aversion will invest...
Table 8. This Table describes behavior of the optimal portfolio vs. equal weighted portfolio for $\rho = 2$

<table>
<thead>
<tr>
<th></th>
<th>AA-AAA</th>
<th>BBB-A</th>
<th>HYBB</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average Optimal CE</td>
<td>104.06%</td>
<td>104.56%</td>
<td>105.92%</td>
</tr>
<tr>
<td>Average EW CE</td>
<td>103.65%</td>
<td>104.00%</td>
<td>104.87%</td>
</tr>
<tr>
<td>Average Difference</td>
<td>0.41%</td>
<td>0.55%</td>
<td>1.06%</td>
</tr>
<tr>
<td>Max CE Difference</td>
<td>3.49%</td>
<td>4.95%</td>
<td>10.99%</td>
</tr>
<tr>
<td>Minimal CE Difference</td>
<td>0.04%</td>
<td>-0.45%</td>
<td>-0.39%</td>
</tr>
<tr>
<td>Average Out Performance</td>
<td>0.42%</td>
<td>0.56%</td>
<td>1.13%</td>
</tr>
</tbody>
</table>

proportionally less of their assets into the risky asset. Another way to see is to compare average weight allocation in optimal portfolios with different risk aversions. For example the average weight that is invested in the high yield bonds rated BB is 64%, while the average weight for the log investor is 96%. We calculated the break even risk aversion for each bucket, that is $\rho$ such that on average an investor with this risk aversion coefficient will be indifferent whether to invest in equally weighted portfolios or in an optimized portfolio corresponding to this $\rho$. The last table presents results of this calculation:

Table 9. This is the table describing the break even risk aversion for each rating bucket

<table>
<thead>
<tr>
<th>Rating</th>
<th>Break Even Risk Aversion Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>AA-AAA</td>
<td>5</td>
</tr>
<tr>
<td>BBB-A</td>
<td>5</td>
</tr>
<tr>
<td>BB</td>
<td>3</td>
</tr>
</tbody>
</table>

The table implies that an investor with a risk aversion of 5 invests equally in risky bonds and the corresponding treasury index because on average this is his optimal portfolio. Recall that on average an estimated risk aversion of a real world utility investor is in the range between 2-5. Our table implies that many investors won’t need to perform optimizations but invest using simple rules because on average portfolios created by these rules will be optimal. This maybe a way to reconcile the apparent gap between optimal portfolios and simple rule allocation portfolios. Note that the break-even risk aversion drops as the riskiness of the asset grow. A lower risk aversion implies an investor that is willing to take more risk. These investors invest a larger proportion of their wealth into riskier assets.

As far as we’re aware the concept of break-even risk aversion didn’t appear in the literature before and it may be an interesting topic for investigation for other asset classes.

7. Constant Maturity Indices - CMI

In this section we consider a problem of one risky asset versus a risk-less asset. The risky asset doesn’t have an expiry or a finite maturity date and for bond indices

\[3\text{The C-CCC will be addressed separately.}\]
this implies that they are being constantly rolled forward from one period to the next. Optimization problems for constant maturity bond indices are equity like optimization problems therefore, we included the equity index for comparison.

We consider the following categories:

- Constant maturity index of corporate bonds with 1-3 years maturity range. The rating of the bonds is: AA-AAA
- Constant maturity index of corporate bonds with 1-3 years maturity range. The rating of the bonds is: BBB-A
- Equity index represented by the SP 500 index (from 1996-2013-the same window as the bond indices)

For each of those indices we produce:

- 1 year predicted returns as an average of the annualized monthly returns of the previous 24 months (\( \mu \))
- 1 year approximation of standard volatility based on the rolling window of 24 months prior to the optimization date (\( \sigma \))
- 1 year treasury return based on the yield of a HTM treasury (1-3) years. (we consider this as a risk free rate consistent with the previous problem.)

Using our estimated parameters we form optimized portfolios using Markovitz’s framework. More precisely given a risky asset with expected return \( \mu \) and standard deviation \( \sigma^2 \) we allocate \( w_1 = \frac{\mu - r}{\sigma^2} \) to the risky asset and \( 1 - w_1 \) to the risk less asset represented by the yield of short term treasury index. Once we do that we calculate real performance during the year and compare it with equal weighted portfolios on two dimensions:

- out-performance or under performance versus the benchmark (equal weighted portfolios)
- Sharpe ratio calculation as a risk adjusted criteria.

The table below describes the statistical properties of the allocation to the risky weight.

<table>
<thead>
<tr>
<th></th>
<th>Average</th>
<th>Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA-AA</td>
<td>55.10</td>
<td>42.60</td>
</tr>
<tr>
<td>BBB-A</td>
<td>50.45</td>
<td>40.90</td>
</tr>
<tr>
<td>Equities (SP 500)</td>
<td>2.41</td>
<td>4.28</td>
</tr>
</tbody>
</table>

Comparing the weights to the HTM log portfolio weights (Table 6) we observe that the allocations are more extreme than in the hold to maturity problem. This happens because of the statistical nature of the CMI corporate index. These are the statistical properties of the treasury and corporate index: Calculating expected weights using the formula we discover that the expected investment in the risky asset is around 3800%. This is compatible with the out of sample weights created on a rolling basis in the table above. On a more qualitative basis we see that the volatility of the risky weights is low while the expected return is relatively high and this provides an incentive to invest huge sums of money into the risky asset. The next graph shows the evolution through time of the optimized portfolios.
Table 11. Statistical properties of the AA-AAA CMI index and HTM treasury index

<table>
<thead>
<tr>
<th></th>
<th>Corporate Index</th>
<th>Treasury Index</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average Monthly Return</td>
<td>0.59%</td>
<td>0.27%</td>
</tr>
<tr>
<td>Standard Monthly Deviation</td>
<td>0.90%</td>
<td></td>
</tr>
</tbody>
</table>

Figure 5. This graph describes the optimized portfolio weights for CMT indices from 1998-2013

This graph shows the extreme weights that are predicted by the usual Markovitz framework. Our next task is to compare the statistical properties of these portfolios versus equal weight naive allocation. We use Sharpe ratio measure to perform this comparison. The statistical properties of the out of sample performances and their Sharpe ratios are given in the table below

Table 12. Statistical properties of Markovitz portfolios for CMT indices with different rating buckets

<table>
<thead>
<tr>
<th></th>
<th>AA-AAA</th>
<th>A-BBB</th>
<th>S&amp;P</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average Performance (Optimized)</td>
<td>52.92%</td>
<td>52.06%</td>
<td>-9.00%</td>
</tr>
<tr>
<td>Average Sharpe Ratio (Optimized)</td>
<td>0.35</td>
<td>0.38</td>
<td>N/A</td>
</tr>
<tr>
<td>Average Equal Weighted</td>
<td>4.00%</td>
<td>4.00%</td>
<td>4.00%</td>
</tr>
<tr>
<td>Average Sharpe Ratio (Equal Weighted)</td>
<td>2.66</td>
<td>1.75</td>
<td>0.48</td>
</tr>
</tbody>
</table>

For the rated bond buckets the out performance versus the equally weighted portfolio is significant but, it performs poorly on the risk adjusted performance metric. Indeed, for the rated investment grade cohorts we see that Sharpe ratios of equally weighted portfolios is superior to the ratios of the optimized portfolios. The most remarkable feature appears in the equity index. For the window we selected (1996-2014) the optimized equity portfolio has negative performance on
average. This isn’t surprising because of the massive losses that equity portfolios sustained during the crisis. In the out of sample performance it’s the failure to predict equity returns (based on historical averages for the window we selected) that leads to this outcome. The other interesting implication seen from this table is the stable return profile of equally weighted portfolios across the different multiple asset classes we selected. The average performance of equally weighted portfolio for the historical period we selected is stable across different risky assets and amounts to 4% – 5% per annum on average.

Finally let us remark that the high leverage involved in the optimal portfolios construction for constant maturity indices makes the suggested portfolios challenging to implement. Inspecting the table we see that these portfolios are constructed using a borrowed money of approximately 98%. From the point of view of the borrower it implies that a drop of 0.5% will probably force him to liquidate the portfolio or to put more collateral to sustain the trade (recall that for constant maturity indices we don’t have any definite time horizon for which the trade will terminate.) This might be too risky for an average investor or a mutual fund manager because of loss aversion (though the loss of more than 2% on a monthly basis occurred only once in the last 20 years. The average amount of the collateral the borrower supplies is 20%. That is borrowing 98% units of currency amount putting up 19.6% units upfront. This makes the suggested portfolios for the fixed income constant maturity indices a highly challenging exercise.

7.1. Break Even Risk Aversion. For HTM strategies we defined a break even risk aversion for equally weighted portfolios to be the risk aversion coefficient \( \rho \) such that the optimal portfolio is the equally weighted portfolio. If \( \gamma \) is the risk aversion than the break even risk aversion equals:

\[
\gamma = \frac{2(\mu - r)}{\sigma^2}
\]

Estimating the \( \mu \) and \( \sigma \) in our time series of total return of the indices we arrive at the following results: Comparing our results with log power utility optimization we see that in the case of constant maturity the break even for bonds is in line with risk aversions obtained from the equity premium puzzle. On the other hand the break even for equities is close to the break even risk aversion that is implied by HTM bonds. Thus, an investor that has a realistic risk aversion between 3 and 5 may not care whether he invests in an optimized portfolio or equal weighted portfolio simply because those portfolios are the same on a long term average. We mentioned that the result we obtain may serve as one approach to reconcile the conflict between equal weighted and optimized portfolios. The conflict doesn’t exist because an investor that uses standard tools to predict returns and volatilities ends up with
the equal weighted portfolios as long term average optimal portfolio. Therefore he will invest in portfolio that is defined by simple heuristics because it’s the optimal long term portfolio in any case. Here we can also see a crucial difference between HTM strategies and constant maturity strategies for bonds since for such strategies the break-even risk aversion are not realistic. This serves as an indirect evidence that the appropriate optimization strategies for bonds are HTM strategies rather than the practice that exists today of constant maturity.

The result we obtained for equities is partial because we considered only one risky asset while the allocation problems in equities require multiple investment decisions. We will further investigate break-even aversions for multi-asset portfolios in future work.

8. Multi Asset Allocation - Introduction

In the previous sections we addressed the allocation of HTM for two assets. That is, generic bonds represented by short term corporate indices and treasuries. In this section we investigate a multi-asset allocation. We consider an allocation of \( N \) bonds in conjunction with a treasury allocation and we compare the performance of the portfolios we obtain with equal weighted portfolio allocation which is our benchmark.

To accomplish this we need the following ingredients:

- Data about returns or historical yields of corporate bullet bonds
- Default model for different corporate bonds
- An algorithm to solve the problem numerically (since it is unlikely we will obtain an analytical solution for the optimization problem)

We explain the data source and the model to obtain the optimized weights. Next, we present results and some sensitivity analysis versus parameters of our model.

8.1. Bond Data. The bond data is based on the Merrill Lynch index rated A-AAA with maturity less than one year. The constituents of this index are available from 2006 and we have seven years of data (2006-2013). The yields and the option adjusted spreads are available for the separate bonds that are in the index. The rules of the index include various maturities from a month to a year. We created a subindex with the following rules:

- If an index member is highly rated preferred stock or a bond with an embedded call dates less than a year from a given date we remove it from our subindex. Thus, we include only corporate bullet bonds in our subindex.
- If a bond has less than 11 months maturity from a given date we remove it from our subindex.

The final data set contains bonds that have:

1. rating of A-AAA
2. Maturity range between 11 and 12 months

We perform this process on a monthly basis and for each month for our period we created an initial set of bonds.

From this list of bonds we select bonds on which we perform the actual optimization. As we like to have portfolios that in theory can be implemented in reality we were interested in the most liquid instruments. Our measure for liquidity was the total amount outstanding of the issue we like to optimize. The bonds with
the biggest total amount outstanding are usually the bonds with the largest market value among the index constituents. Thus, for our optimization universe we selected the 3 biggest bonds with the largest market value among the initial set of bonds we selected. We optimized on the latest group of bonds we created on a monthly basis.

9. Common Default Model

Bonds may or may not default simultaneously. We selected Vasicek’s model to describe bond default dynamics. This model is utilized by various institutions for synthetic credit product valuation. We use it to simulate joint bond default scenarios.

Let $B_1, \ldots, B_N$ be corporate bonds. We assume that underlying companies of these bonds have default probabilities $p_1, \ldots, p_N$. Now assume that each company value is given through the following:

$$V_i = \sqrt{\sigma Y} + \sqrt{1-\sigma} Z_i$$

where $Z_i, Y$ are standard normally distributed independent variables. Thus, the value of the underlying companies is normally distributed and the correlation between any of the companies equals to $\sigma^2$. We assume that bankruptcy for each company occurs when the value of the company $V_i$ falls below a certain threshold $B_i$. If the default probability of company $V_i$ is $p_i$ then $B_i = N^{-1}(p_i)$. We simulate multiple pathes of companies using the last equation and in the scenarios we create bonds may default or not default together.

9.1. Model Calibration. We explain how we calibrate model parameters for the bond universe we selected. As our index is A-AAA we select the long term default probability of 0.1% for all bonds. Our next task is to estimate the correlation of defaults. Because the default for A-AAA bonds are rare it is not possible to perform historical analysis and find the real world correlation between defaults. Even if could perform such an analysis the confidence intervals for these correlations would be big and they will be highly uncertain. As an alternative we rely on correlations that are implied by the market. These correlations are derived from Vasicek model. Structured product traders use them to hedge and derive the price of synthetic CDOs. These are products where protection is bought not on a single name but on certain slices of the capital structure in a portfolio that is represented by credit default swaps of the underlying index names. The pricing of these is performed in the risk neutral setting and the correlation in the Vasicek model plays a similar role to volatility in the option price theory. The default correlation from this is a risk neutral correlation and not a real world correlation. In order to mitigate this we perform a sensitivity analysis on optimal portfolio weights as a function of the underlying correlation. If portfolio weights don’t change dramatically we conclude that the implicit assumption we make (real world correlation equals to risk neutral correlation) is robust against possible errors in the correlation estimation that may result from this assumption. Our base case is $\sigma = 0.4$. This is average standard correlation as derived from the equity tranche of synthetic CDOs.

9.2. Setting the optimization and numerical algorithm for solution. We set the optimization and solve it numerically as follows: Assume we simulated $L$ scenarios for $N$ bonds $B_1, \ldots, B_n$ with yields $y_1, \ldots, y_n$. Assume we are investing $w_1, \ldots, w_n$
of each security. Let \( m \) be the number of scenarios we simulated. For \( j \)-th scenario let the outcome for bond \( B_i \) be \( L_{ij} \). And \( L_{ij} \) is \( 1 + y_i \) if no default occurred in this scenario and \( w_i R_i \) otherwise. We need to optimize the following log utility problem:

\[
\frac{1}{m} \left( \sum_{i,j=1}^{n,m} \ln \left( w_i L_{ij} + \left( 1 - \sum w_i \right) \times (1 + r) \right) \right)
\]

Derive versus \( w_i \) to obtain the following system of equations:

\[
\frac{1}{m} \sum_{j=1}^{m} w_i (L_{ij} - (1 + r)) = 0
\]

Let \( S_i \) denote the \( i \)-th equation. We replace this system equation with a single equation: \( \sum_{j=1}^{n} S_j^2 = 0 \) This is the optimization equation we solve using excel solver. We report the results of the optimization performance versus \( \frac{1}{N} \) in the next subsection.

9.3. **Optimization Results** \( N = 3 \). Let us display the results of the optimization for \( N = 3 \) assets. Our assumptions are:

1. Default correlation based on Vasicek model: \( \sigma = 0.4 \).
2. Probability of default for a single bond is \( 0.1\% \).
3. We selected 3 bonds with the biggest weight in the synthetic index of the bullet bonds we created.
4. We simulated 1000 scenarios of defaults using Vasicek model for these bonds.

The tables below display:
- statistical properties of the optimized portfolio weights time series we obtained.
- performance of our optimized portfolio versus the \( \frac{1}{N} \)

**Table 14. Statistical properties of the optimal portfolio weights for 3 assets.**

<table>
<thead>
<tr>
<th></th>
<th>Asset 1</th>
<th>Asset 2</th>
<th>Asset 3</th>
<th>Treasury</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average Optimal Weight</td>
<td>1.03</td>
<td>1.48</td>
<td>1.43</td>
<td>-2.94</td>
</tr>
<tr>
<td>Standard Deviation of Weights</td>
<td>0.64</td>
<td>0.33</td>
<td>0.61</td>
<td>1.14</td>
</tr>
</tbody>
</table>
Table 15. Statistical properties of the out-performance of the multi-asset optimized portfolio versus equal weighted benchmark

<table>
<thead>
<tr>
<th>Average Performance Optimized 6.80%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average Performance Equal Weighted  2.73%</td>
</tr>
<tr>
<td>Average Out Performance (Annualized) 4.08%</td>
</tr>
<tr>
<td>Max Out-performance 22.16%</td>
</tr>
<tr>
<td>Min Out-performance 0.41%</td>
</tr>
<tr>
<td>Stdev 3.97%</td>
</tr>
<tr>
<td>Optimal CE 2.90</td>
</tr>
<tr>
<td>Equal CE 2.79</td>
</tr>
<tr>
<td>Kurtosis(Optimal) 4.51</td>
</tr>
</tbody>
</table>

The tables above show that we have a significant out-performance of the optimized portfolio versus the equal weight portfolio. The optimizer shortens the treasury index while buying the corporate bonds. The major out-performance came in the crisis period because at that time the spreads of the corporate bonds were the most elevated and thus, the optimized portfolio had the most out-performance at that period. Further the average out-performance is more significant than in the single risky asset case. This isn’t surprising because the shortening of the risk-less asset in our case is more significant than in the case of a single asset. Indeed, The average weight for the treasury asset for the optimization problem with a single risky asset was $-20\%$ and here we see a treasury shortening of almost $300\%$. The shortening of the treasury implies a portfolio with an approximate leverage of $1:4$. This is within of realms of realistic portfolios that can be implemented in reality. Looking on the CE of the equal weighted versus the optimized portfolio observe that the difference isn’t dramatic though the average CE of the optimized portfolio is consistently higher.

9.4. Correlation Sensitivity Analysis. To investigate the dependency of the optimized portfolio on a correlation we performed the optimization analysis with different default correlations numerically to investigate whether results change significantly if correlations are perturbed.

Table 16. Optimal allocation of portfolios where different correlations are employed for the Vasicek model we assumed above.

<table>
<thead>
<tr>
<th>Correlation</th>
<th>10.00%</th>
<th>20.00%</th>
<th>30.00%</th>
<th>40.00%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average Weight Asset 1</td>
<td>126.00%</td>
<td>81.00%</td>
<td>103.00%</td>
<td>103.53%</td>
</tr>
<tr>
<td>Average Weight Asset 2</td>
<td>130.00%</td>
<td>147.00%</td>
<td>148.00%</td>
<td>147.66%</td>
</tr>
<tr>
<td>Average Weight Asset 3</td>
<td>117.00%</td>
<td>115.00%</td>
<td>115.00%</td>
<td>142.50%</td>
</tr>
<tr>
<td>Average Treasury Weight</td>
<td>-273.00%</td>
<td>-243.00%</td>
<td>-267.00%</td>
<td>-293.69%</td>
</tr>
<tr>
<td>Average Optimized Performance</td>
<td>6.90%</td>
<td>6.80%</td>
<td>6.60%</td>
<td>6.80%</td>
</tr>
<tr>
<td>Average Optimized Equal Weighted</td>
<td>2.73%</td>
<td>2.73%</td>
<td>2.73%</td>
<td>2.73%</td>
</tr>
<tr>
<td>Optimal CE</td>
<td>2.91</td>
<td>2.91</td>
<td>2.90</td>
<td>2.91</td>
</tr>
<tr>
<td>Equal CE</td>
<td>2.79</td>
<td>2.79</td>
<td>2.79</td>
<td>2.79</td>
</tr>
</tbody>
</table>
The table shows that for different correlations, the optimized weights that are close to the weights for the original correlation. It seems that correlations don’t have a major impact on the optimized weights. Thus, there is an annual out-performance of around 4% of optimized portfolios versus equal weighted portfolios we have. The stability of the weights implies that correlation assumption may not be critical to the construction of optimized portfolios for small amount of bonds using the method we suggest.

10. APPLICATION TO LIABILITY MANAGEMENT

In the previous section we investigated the behavior of the optimal HTM portfolios and compared it to equal weighted ones. In this section we show how we can apply our optimal portfolio to potentially improve liability management for retirees and possibly insurance companies.

Consider a loan holder that has a loan that expires in 1 year. The holder of the loan can be bank and the loan is a 1 year certificate, or, an insurance company whose liability profile is a treasury bond 1 year from now. The loan holder has several options and instruments to invest in.

His options are:

(1) Short Term Corporate Debt (represented by short term indices like Barclays or Merrill)
(2) Optimized portfolio from last section
(3) Liability portfolio

All indices available for investment and managed by the main providers (like Barclays, Merrill Lynch) are constant maturity indices where bonds are purchased or leave the index based on maturity, sector and liquidity constraints. As no HTM indices are available we compare the first two strategies and find out whether an optimal portfolio strategy can beat a simple purchase of the corporate index that is CMT. We buy a corporate index because we like to make a profit above the liability loans we collected. The table below documents performance of the strategies versus the liability benchmark (which is 1 year treasury with the average yield of 0 – 1 treasury index). The first row refers to the performance of the stand alone strategies while the other rows examine the out-performance (or under performance) of the strategies versus the benchmark (liability). In general if \( x_i, y_i \) are the final wealth of the portfolio and the benchmark the rows examine the series:

\[
\log(x_i) - \log(y_i).
\]
Table 17. Result for the performance of three strategies: The first invests in the optimal portfolio from last section, the second invests in the corporate Merrill Lynch index short term index and the third invests in the liability.

<table>
<thead>
<tr>
<th></th>
<th>Optimized</th>
<th>Corporate Index</th>
<th>Liability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average Performance (Stand Alone)</td>
<td>6.91%</td>
<td>3.68%</td>
<td>1.94%</td>
</tr>
<tr>
<td>Certainty Equivalence (vs. The benchmark)</td>
<td>102.1%</td>
<td>100.7%</td>
<td>100.00%</td>
</tr>
<tr>
<td>Max CE (vs. benchmark)</td>
<td>110.00%</td>
<td>103.89%</td>
<td>100%</td>
</tr>
<tr>
<td>Min CE (vs. benchmark)</td>
<td>100.20%</td>
<td>99%</td>
<td>100%</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>4.51</td>
<td>-1.23</td>
<td>0.45</td>
</tr>
</tbody>
</table>

Figure 6. This graph the rolling monthly annual performance of the portfolios in the text 2006-2013

The table and the graph show that the optimal portfolio we construct outperforms the corporate constant maturity index and the liability synthetic index we constructed. The maximal out-performance occurs during the crisis period 2007-2008. While the corporate index under-performs the liability component during this period, optimized portfolios out-perform the liability during this period. This is the result of HTM strategy. Bonds that are issued during this period have a higher yield and if they aren’t defaulted they mature and get the principal back. Hence they are immune against the price fluctuations that the corporate index has because of continuous bond trading.

The other surprising feature of the results above is the consistent out-performance of the optimal portfolio versus the corporate constant maturity index. During the period we examined the optimal portfolio didn’t have even one month of under-performance versus the corporate index. This can be attributed to the relative high leverage of the optimal portfolio. This results in a high yield of the optimized portfolio that in turn boosts its performance significantly.

The results of this may have an application to liability management. The main strategies that market participants use to out-perform liabilities is to invest in
corporate indices that are usually constant maturity. This may run into the risk of under-performing the corresponding liability stream in times of crisis.

The analysis we carried above suggests an alternative way to manage liabilities using optimized portfolios of corporate bonds. These optimized portfolios may be more immune to an economic downturn because of the pull to par effect and they also deliver a higher out-performance compared to the constant maturity corporate index. The weakness of the optimized portfolios is their leverage and the relative high kurtosis compared to the corporate index and the liability benchmark. This may deter investors to implement the optimized weights as described above. However, the optimized weights of the portfolio can be used as an initial benchmark and the weights of the individual bonds (and the bonds themselves) can be selected based on the risk appetite and recommendations of the credit analysts and portfolio managers.

Let us briefly discuss liquidity issues associated with these portfolios. In the vast majority of cases liquidity issues in any market arise when there are many sellers and too few buyers. To clear the prices and entice the buyers the sellers are forced to lower the price dramatically to get rid of their inventory. The optimized portfolios are long corporate issues and hence when liquidity crisis occurs it will be advantageous for buy and hold portfolios as they buy and hold until maturity without being involved in the active trading. The other issue that liquidity can cause is the shorting of the treasury. However as the treasury has a short horizon to maturity (in our case) and as its duration is around 3 years conclude that the price loss of the treasury will be between 3\(^{-}-\)4\(^{-}\) (this occurred in the 2008 crisis.) as the negative weight of the treasury is around \(-3\) units conclude that at most our portfolio loses \(-12\%^{-}\). This loss is within the realistic boundaries and shouldn’t produce an issue for the optimal portfolio construction.

11. Summary and Discussion

In this paper we investigated optimization properties for bonds. The major difference between the optimization problems we consider and the standard optimization problems is that we assume a finite maturity horizon and a concrete strategy that is hold to maturity. This paper implies that the finite maturity requirement may be an additional important condition for bonds that changes results sometimes significantly compared to other asset classes as stocks. We have evidence that the classical optimization techniques most notably the utility function approach that have been shown to have major challenges to boost performance of stock portfolios versus simple rule benchmarks may be better suited for finite maturity instruments with low probability of default. This occurs because the relative stability of bond default probability. Hence we can predict bond returns with a bigger confidence than stock returns. This eliminates a major hurdle from the optimization process and allows the planner to focus on optimization techniques in order to improve portfolio performance.

For bonds, returns of such strategy are not normally distributed, Hence Markovitz’s mean variance classical approach is inadequate. We consider a log utility optimization instead in this paper. In the first part we analyzed optimized portfolios for one risky asset, where we obtain an analytical solution for the log utility and in a
multi-asset case where we solve the problem numerically. Our next step is to compare the performance of the optimal portfolios we obtained with the corresponding performance of two types of portfolios:

- equally weighted portfolios (single and multiple assets)
- index based portfolios relative to the optimized multi-asset portfolios
- optimal CMT portfolios based on Markovitz mean variance framework.

We compared the performance and concluded that for all rating buckets that are rated BB and above the optimal portfolio outperforms a portfolio that is 50% in the risky asset and 50% in the risk-less asset. The explanation we suggest to this phenomena in the paper is straightforward: For HTM optimized portfolios the only source of risk is the default risk. For higher rated bonds (BB and above) default rates tend to be stable, hence they are easier to predict. Moreover the real default rate is low historically and thus, portfolios obtained using our formula tend to be mildly levered. This assures that they out perform equally weighted portfolios because the performance tend to be in line with the predicted performance that relies only on the current yield of the portfolio.

This phenomena is prevalent for risk aversion coefficient which is different than 1. In an attempt to quantify the out-performance dependence on risk aversion coefficient we defined the break even risk aversion ($\gamma_{br}$). An investor with a certain $\gamma_{br}$ is indifferent whether to invest in an optimal portfolio or in an equally weighted portfolio on long term average. The calculation of the break even risk aversion coefficient implies that these are realistic risk aversions and therefore large swaths of population will view equal weighted HTM portfolios as optimal portfolios.

An interesting observation is the break even risk aversion for equity using Markovitz analysis. On average we discovered that $\gamma_{br}$ is around 4. This is a realistic risk aversion (the implied equity premium risk aversion are much higher.) and hence for many equity investors should consider the strategy suggested by Uppal, De Miguel, Garlappi (2009) as the equal weighted investments will be an optimal one. This may be a possible way to resolve the debate that exists between the supporters of optimization and supporters of investing based on simple rules.

The next step in our investigations was to consider a multiple asset allocation for HTM strategy. We compared it with two strategies:

- Equal weighted strategy
- Index strategy - buying investment grade short term bonds

We stress that we compare with a constant maturity index in the second strategy because there are no indices available that are invested in HTM like strategy.

While the equal weighted strategy under performed our optimized portfolio we discovered that our strategy out-performed the CMT corporate bond index also. This is surprising because indexation is widely considered to be the recommended strategy for pension planners and liability management investors. However, optimized portfolios have a leverage built in hence, if they don't default they are almost guaranteed to out perform a non-levered portfolio. We consider the latest conclusion as the main result of our paper and arguing that details count and incorporating finite maturity into optimizations may improve liability management.

The latest result opens some interesting route for investigations that we plan to pursue in the future: First we considered only short term horizon in our portfolios. More precisely the holding period of our portfolio was only one year. It may happen that in longer time periods the strategy we advocate in this paper may fail because
of the difficulty to predict defaults. On the other hand from the historical default table we observe that even for longer time horizons the default rate is remarkably stable as evident inspecting the tables.

A question which isn’t less important is to try and implement a dynamic strategy for HTM portfolios in the spirit of Merton. When bonds mature Merton’s framework fails because volatility tends to fall to 0. Is it possible to solve the problem of optimal dynamic allocation assuming finite maturity of instruments and jump to default?  

Finally a practical matter of this research is important. It is interesting to observe buy and hold strategies in real time and see whether they will out-perform CMT index based strategies in the future as well.

References


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4Recently the author obtained results in this direction. The paper is in the works