Prudence with Multiplicative Risk

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Abstract
In this paper we analyze the conditions under which the presence of a multiplicative background risk induces a more "prudent" behavior. We show that the results from Kimball (1990) concerning the convexity of the marginal utility are no longer sufficient with multiplicative risk. An agent is multiplicative risk prudent when the coefficient of relative prudence is greater than two. We introduce the concept of quintessence in order to guarantee the decrease of relative temperance. Both decreasing relative prudence and decreasing relative temperance are sufficient to guarantee more "prudent" behavior with a multiplicative risk-prudent agent. Nevertheless, the convexity of the relative prudence combined with the decrease (respectively increase) of the relative prudence implies more prudent behavior when the agent is multiplicative risk prudent (respectively risk imprudent). Surprisingly, the derived utility function inherits some properties of the direct utility function and the presence of the multiplicative background risk preserves comparative risk-prudent behavior under certain conditions. Our results apply to savings and portfolio selection. For example, a risk-prudent agent will lower his or her willingness to hold risky assets when he or she faces a multiplicative background risk if his or her relative risk aversion is convex and decreases.

Keywords: additive background risk, multiplicative background risk, risk vulnerability, risk aversion, prudence, temperance

JEL: D81, G11

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1. Introduction
This paper addresses the question of whether the presence of a multiplicative background risk induces more risk-prudent behavior. What is the effect of a unitary-mean multiplicative background risk on the feeling of prudence? Intuitively, the presence of a background risk will affect behavior. In fact, Franke, Stapelton and Schlesinger (2006) develop the concept of multiplicative risk aversion and they show that the effect of the background risk depends on the type of risk with which the individuals are confronted. Attitudes towards risk differ depending on whether the agent faces additive risk or multiplicative risk. However, they undertake their analysis by focusing on the alteration of risk aversion. Here, we deal with risk prudence. That is, we focus on situations, such as savings or insurance, in which marginal effects are great. Prudence is related to three concepts: precautionary savings, downside risk and preference for positive skewness.

They has been special attention on additive risks in multi-risk problems that have been studied only recently. Kihlstrom, Romer and Williams (1981), Pratt and Zeckhauser (1987) and Pratt (1988) make significant contributions. Kimball (1990) defines the concept of prudence or additive risk prudence, which implies that the presence of some additive background risk increases the optimal savings rate. This concept is equivalent to the non-negativity of the third derivative of the utility function or to the convexity of the marginal utility. Kimball (1993) defines, in an expected utility framework, the property of standard risk-aversion which implies that the presence of some background risk reduces the optimal investment in a risky security with an independent return. Eeckhoudt, Gollier and Schlesinger (1996) study the effect of independent exogenous risk on the optimal risk-taking behavior towards an endogenous risk. They examine conditions on preferences under which some changes in the distribution of the background wealth entail more risk-averse behavior towards endogenous risk. Eeckhoudt and Kimball (1992) study how a background risk affects optimal demand for insurance against an independent or dependent insurable risk. Eeckhoudt, Gollier and Schlesinger (1996), Gollier and Pratt (1996), Kimball (1993) and Pratt and Zeckhauser (1987), among others, show that, under fairly general conditions on the utility function, investors reduce their holdings of risky assets when facing an increase in the background risk. All these studies implement the derivatives of higher order of the utility function. Caballé and Pomansky (1996) present the notion of mixed risk aversion in order to sign the derivatives of the utility function and their results allow us to reconsider the relationship between those derivatives and the moments (variance, skewness, kurtosis). Eeckhoudt and Schlesinger (2006) introduce a new but a more general concept, the concept of risk apportionment of any order. This new concept allows us to recover all the former concepts in risk theory and also permits us to characterize those concepts with preference relation over specified lotteries which consist of two basic dislikes: a certain reduction in wealth adding a zero-mean independent noise random variable to the distribution of wealth. All these papers deal with additive risk. They pay little attention to multiplicative risk. Franke, Stapelton and Schlesinger (2006) are the first to focus on the effect of a multiplicative risk but they restrict themselves to the feeling of risk aversion. They find out conditions under which the presence of the multiplicative background risk leads to more cautious behavior. Our aim in this paper is to analyze the more prudent behavior of an agent facing a multiplicative risk. Under which conditions does the agent behave more prudently way in the presence of the multiplicative background risk? And what are the implications on savings decision and risky assets holdings?

Dealing with multiplicative risk, we show that the convexity of the marginal utility function is no longer sufficient to guarantee risk-prudent behavior in the presence of a multiplicative background risk. We must compare the coefficient of relative prudence with a benchmark value equal to two. More precisely, an agent is said to be multiplicative risk prudent if he/she increases his/her savings when facing multiplicative background risk. We also introduce the concept of risk apportionment of order five which we call quintessence to guarantee the decrease of the feeling of temperance. Quintessence is to temperance as temperance is to prudence. It appears that the results obtained with additive risks are ill suited to multiplicative risks.
Our paper contributes to the literature devoted to the attitude toward risk and the decision making in presence of risks in several ways. First, we provide a comprehensible definition of prudence when facing a multiplicative risk to complement the well known definition of Kimball (1990). Second, we give the conditions under which in the presence of a multiplicative background risk an agent behaves in a more risk-prudent manner. That is, we compare the degrees of relative prudence in the presence and in the absence of the multiplicative risk. Third, we show the link between decreasing relative prudence of the derived utility function and decreasing relative prudence of the direct utility function. Fourth, we point out conditions under which the comparative risk prudence order is preserved. Fifth, we consider savings demand and portfolio selection and we apply our results.

The paper is organized as follows. Section 2 presents the concepts governing the attitudes toward risk. Section 3 is devoted to the definition of multiplicative risk prudence. In section 4 we introduce the affiliated utility function. The next section gives the conditions under which an agent facing a multiplicative background risk behaves in a more risk-prudent manner. Section 6 deals with the preservation of the decrease of relative prudence. In section 7 we analyze the preservation of the ordering by risk prudence with multiplicative background risk. Sections 8 and 9 apply our result to savings and portfolios. The last section concludes.

2. From risk aversion to quintessence

Modelling the behavior of investors facing uncertainty is generally done with the concept of risk aversion, which is related to the concavity of the utility function. The degree of risk aversion is given by:

$$ R_u^a(w) = -\frac{u''(w)}{u'(w)} $$

An individual will be risk averse if the second derivative of the utility function is negative ($u'' \leq 0$).

However, the concept of risk aversion has some limits and it is not appropriate when dealing with decision-making under uncertainty when the uncertainty affects marginal utility rather than utility. To solve this problem, Kimball (1990) uses the notion of absolute prudence to measure the propensity to prepare and to forearm oneself for uncertainty. The degree of absolute prudence is given by:

$$ P_u^a(w) = -\frac{u'''(w)}{u''(w)} $$

An individual is prudent or additively risk-prudent when the third derivative of his or her utility function is positive ($u''' \geq 0$).

These two notions of absolute risk aversion and absolute prudence are very rich. When taken together, they correspond to the notion of standardness, which is sufficient and necessary to see an increase in the demand for a second independent risk due to a risk that increases the marginal utility function. Decreasing absolute prudence makes it necessary for temperance to be higher than prudence, where the coefficient of absolute temperance is given by:

$$ T_u^a(w) = -\frac{u^{(4)}(w)}{u'''(w)} $$

An individual is temperate when the fourth derivative of his or her utility function is negative ($u^{(4)} \leq 0$). Temperance describes the desire to totally reduce the risk exposure. That is, the agent reduces an endogenous risk when an exogenous risk increases.

Nevertheless, absolute risk aversion and absolute prudence sometimes fail to explain the behavior of an investor facing several sources of risk and in the event of a deterioration of those risks. Other notions, such as proper risk aversion and risk vulnerability can be used. Pratt and Zeckhauser
(1987) devise the concept of proper risk aversion to characterize utility functions such that an undesirable risk can never be made desirable by the presence of an independent undesirable risk. A risk is said to be undesirable if it decreases the expected utility. The utility functions whose successive derivatives alternate in sign exhibit proper risk aversion. Gollier and Pratt (1996) introduce the concept of risk vulnerability to model the fact that an investor reduces his or her risky assets when facing an increase in an independent unfair background risk. That is, an undesirable risk can never be made desirable by an independent unfair risk.

In our study, the coefficient of absolute temperance will be made to decrease with wealth. This condition will imply that the coefficient of absolute temperance is lower than a new coefficient, which we call quintessence given as follows:

$$Q^A_w = -\frac{u^{(5)}}{u^{(4)}}(w)$$

An individual exhibits quintessence when the fifth derivative of his or her utility function is positive ($u^{(5)} > 0$). Eeckhoudt and Schlesinger (2006) call this coefficient risk apportionment of order 5.

We call this coefficient quintessence because we are dealing with the fifth derivative of the utility function and quinte means five and essence substantial nature. It is precious and we want to protect it by any means. Recall that this word comes from Latin "quinta essentia" which means literally "fifth essence". It is the fifth element after earth, fire, water and air. Therefore, it will come after non-satiety ($u' > 0$), risk aversion ($u'' \leq 0$), prudence ($u''' \geq 0$) and temperance ($u^{(4)} \leq 0$).

Temperance is to quintessence as prudence is to temperance. We can notice that quintessence corresponds to the prudence of the second derivative of the utility function:

$$Q^A_w = -\frac{(u^{(4)})''}{(u^{(3)})''}(w) = P^A_u(w)$$

We can also say that quintessence represents the temperance of the opposite of the marginal utility:

$$Q^A_w = -\frac{(-u^{(4)})'}{(-u^{(3)})'}(w) = T^A_u$$

We get the relative counterpart of each coefficient by multiplying it by wealth. This is important because dealing with multiplicative risk induces the use of relative coefficients:

$$R^A_u(w) = -w \frac{u^{''}}{u'}(w) ; P^R_u(w) = -w \frac{u^{''''}}{u''}(w) ; T^R_u(w) = -w \frac{u^{(4)}}{u'''}(w) \text{ and } Q^R_u(w) = -w \frac{u^{(5)}}{u^{(4)}}(w)$$

We begin by the fact that relative prudence and relative temperance are generally both decreasing and we analyze the main consequences.

Property 1: Decreasing relative prudence implies that relative temperance exceeds relative prudence plus one and decreasing relative temperance implies that relative quintessence exceeds relative temperance plus one.

Proof. The derivative of relative prudence with respect to wealth is:

$$\frac{d}{dw} \left[ -\frac{u^{''}}{u'}(w) \right] = -\frac{u'''(w)}{u''(w)} \left[ 1 + \frac{u^{(4)}(w)}{u^{(3)}(w)} - \frac{u^{(4)}(w)}{u''(w)} \right] = P^A_u(1 - T^R_u + P^R_u)$$

Therefore, decreasing relative prudence is equivalent to $P^A_u(1 - T^R_u + P^R_u) \leq 0$. 


Finally, an increase in wealth leads to a decrease in relative prudence if and only if $1 + P_R^u \geq R_R^u$. The same approach yields the result concerning the decrease of the coefficient of relative temperance.

If we consider the decrease of the coefficients of relative risk aversion, relative prudence and relative temperance we end up with the following property:

**Property 2:** Decreasing relative risk aversion, decreasing relative prudence and decreasing relative temperance imply:

1. $Q_R^u \geq T_R^u \geq P_R^u \geq R_R^u$.
2. $P_R^u \geq 2, T_R^u \geq 3$ and $Q_R^u \geq 4$ if $R_R^u \geq 1$.

**Proof.** Decreasing relative risk aversion implies $P_R^u \geq 1 + R_R^u$ and therefore $Q_R^u \geq 1 + P_R^u$. Finally, decreasing relative prudence implies $T_R^u \geq 1 + P_R^u$ and therefore $Q_R^u \geq T_R^u$.

We get $Q_R^u \geq T_R^u \geq P_R^u \geq R_R^u$.

Assume that $R_R^u \geq 1$ and knowing that $P_R^u \geq 1 + R_R^u$, thus $R_R^u \geq 2$.

And adding the fact that $T_R^u \geq 1 + P_R^u$ gives $T_R^u \geq 3$. Therefore, knowing that $Q_R^u \geq 1 + T_R^u$ leads to $Q_R^u \geq 4$.

It is worth noting that the former property uses the benchmark values of our four coefficients: one for relative risk aversion, two for relative prudence, three for relative temperance and four for relative quintessence. That leads to sixteen classes of utility functions according to the position of the coefficients with respect to their respective benchmark:

$$\text{Class} \{(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \{u \text{ such that } (R_R^u - 1)\lambda_1 \geq 0; (P_R^u - 2)\lambda_2 \geq 0; (T_R^u - 3)\lambda_3 \geq 0; (Q_R^u - 4)\lambda_4 \geq 0\} \text{ where } \lambda_i \in \{-1, 1\}$$

For example, any element of $\text{Class}(1,1,1,1)$ is such that $R_R^u \geq 1, P_R^u \geq 2, T_R^u \geq 3$ and $Q_R^u \geq 4$. That is, any element of this class exhibits simultaneously higher feelings of relative risk aversion, relative prudence, relative temperance and relative quintessence.

With these sixteen classes, we break down the set of well behaved utility functions according to their high or low feeling into the four feelings. High means higher than the benchmark, whereas low means lower than the same benchmark.

### 3. Multiplicative derived utility function

In this section, we present the general model. Let us consider an individual facing a multiplicative risk $1 + \tilde{x}$ (or $\tilde{x}$) such that $E(\tilde{x}) = 0$ and $1 + \tilde{x} \geq 0$ which affects his or her entire wealth and with a well behaved utility function $u \{ (u'w) \geq 0; u''(w) \leq 0; u'''(w) \geq 0 \}$. We define the multiplicative derived utility function associated with $u$ as follows:

$$\tilde{u}(w) = Eu \{ w(1 + \tilde{x}) \}$$

The marginal utility derived from the situation is given by:

$$\tilde{u}'(w) = E \{ (1 + \tilde{x})u' \{ w(1 + \tilde{x}) \} \}$$

The multiplicative risk positively affects the behavior of the agent if any variation of wealth has a greater effect in the presence of the risk or if the marginal utility is higher in the presence of the multiplicative background risk. We have the following definition:
Definition 1: \( u \) is multiplicative risk-prudent if and only if the marginal utility of \( \hat{u} \) is greater than that of \( u \).

This formulation is analogous to that of Kimball (1990), who defines the feeling of prudence in the presence of an additive risk as follows:

\[
\nu(w) = Eu(w + \varepsilon) \text{ and } E(\varepsilon) = 0
\]

According to Kimball (1990), prudence is equivalent to the convexity of the marginal utility in order to guarantee that \( \nu'(w) = Eu'(w + \varepsilon) > u'(w) \). Therefore, an agent is prudent if adding a zero-mean background risk to his or her second-period income leads to an increase in savings. This notion of prudence will be called additive prudence.

An agent will be multiplicatively risk-prudent if multiplying his or her second-period wealth by a unitary-mean background risk increases savings. In this situation, the marginal utility of \( \hat{u} \) is greater than that of \( u \) if and only if

\[
\hat{u}'(w) = E(1 + \varepsilon)u'(w + \varepsilon) \geq u'(w) \]

or if and only if \( \varepsilon' u'(wx) \) is convex in \( x \). The last condition is equivalent to

\[
 Pu_R(w) = -\frac{u''(w)}{u'(w)} \geq 2
\]

Here, we deal with multiplicative prudence.

Let us consider the definition of prudence given by Eeckhoudt and Schlesinger (2006):

Definition 2: “An individual is said to be prudent if the lottery \( B_3 = \{ -k; \varepsilon \} \) is preferred to the lottery \( A_3 = \{ 0; \varepsilon - k \} \), where all outcomes of the lotteries have equal probability, for all initial wealth level, for all \( k \) and for all (zero-mean random variable) \( \varepsilon \).”

In our framework, we consider multiplicative risk rather than additive risk. Therefore, the former definition becomes:

Definition 3: “An individual is said to be (multiplicatively) prudent if the lottery \( B = \{ w - k; w(1 + \hat{x}) \} \) is preferred to the lottery \( A = \{ w - k(1 + \hat{x}); w \} \), where all outcomes of the lotteries have equal probability, for all initial wealth level \( w \), for all \( k \) and for all unitary-mean random variable \( 1 + \hat{x} \).”

In the expected utility framework, the definition becomes:

\[
\frac{1}{2} E(u(w + w\hat{x})) + \frac{1}{2} E(u(w - k)) > \frac{1}{2} E(u(w - k + (w - k)\hat{x})) + \frac{1}{2} E(u(w))
\]

Or equivalently:

\[
E(u(w + w\hat{x})) - E(u(w - k + (w - k)\hat{x})) > u(w) - u(w - k)
\]

We know that:

\[
u(w + w\hat{x}) - u(w - k + (w - k)\hat{x}) = \int_{w-k}^{w} (1 + \hat{x})u'(t + x) \, dt
\]

\[
\text{and } u(w) - u(w - k) = \int_{w-k}^{w} u'(t) \, dt
\]

Therefore, the definition can be expressed as follows:

\[
E \left[ \int_{w-k}^{w} (1 + \hat{x})u'(t + x) \, dt \right] > E \left[ \int_{w-k}^{w} u'(t) \, dt \right]
\]

Thus, \( E \left[ (1 + \hat{x})u'(t + x) \right] > u'(t) \), which is true if and only if \( (1 + x)u'[t + x\hat{x}] \) is convex in \( t \). That is the case when relative prudence is greater than two. We can say that a multiplicatively risk-prudent individual prefers to attach the multiplicative risk to the better outcome.

The definition from Eeckhoudt and Schlesinger (2006) is equivalent to our definition if we replace the additive risk by a multiplicative risk and the non-negativity of the third derivative of the utility function by the fact that the coefficient of relative prudence is greater than two.
This equivalence helps us to point out conditions under which the derived utility function \( \hat{u} \) is multiplicatively risk-prudent knowing that the direct utility function \( u \) exhibits multiplicative risk prudence.

**Property 3:** Consider a well behaved utility function \( u \) which exhibits multiplicative risk prudence, then the derived utility function is multiplicatively risk-prudent if and only if

\[
E(\phi(Z^2)) > E(\phi(Z)) \quad \forall Z, \text{ where } \phi(z) = z' u(wz).
\]

**Proof.** \( \hat{u} \) is multiplicatively risk-prudent if the lottery \( B = [w - k; w(1 + \hat{x})] \) is preferred to the lottery \( A = [w - k(1 + \hat{x}); w] \). That is,

\[
E(\hat{u}(w + wx) - \hat{u}(w - k + (w - k)\hat{x})) > \hat{u}(w) - \hat{u}(w - k) \quad \forall w, k, \hat{x}
\]

Or:

\[
E(u'(w(1 + \hat{x})^2) - Eu[(w - k)(1 + \hat{x})^2]) > Eu[w(1 + \hat{x})] - Eu[(w - k)(1 + \hat{x})].
\]

The last condition yields:

\[
E \left( \frac{1}{w-k} \int_{w-k}^{w} (1 + \hat{x}) u' [t(1 + \hat{x})^2] dt \right) > E \left( \frac{1}{w-k} \int_{w-k}^{w} (1 + \hat{x}) u' [t(1 + \hat{x})] dt \right)
\]

The condition becomes:

\[
E(\phi(Z^2)) > E(\phi(Z)) \quad \forall Z, \text{ where } \phi(z) = z' u(wz).
\]

The derivative of \( \phi \) with respect to \( z \) is given by:

\[
\phi'(z) = u'(wz) + wz u''(wz).
\]

Therefore, the second derivative of \( \phi \) will be:

\[
\phi''(z) = 2wu''(wz) + w^2zu'''(wz) = wu'''(wz) \left[ 2 + (wx) \frac{u''(wz)}{u'''(wz)} \right].
\]

And \( \phi \) is convex if and only if \(- (wx) \frac{u''(wz)}{u'''(wz)} \geq 2 \). That is, the direct utility function is multiplicatively risk-prudent.

We now present some properties associated with the derived utility function and relative to its \( n \)th derivatives. We define the \( n \)th order coefficient of relative risk aversion as follows:

\[
RR_n(w) = \frac{-w u^{(n+1)}(w)}{u^{(n)}(w)}
\]

**Property 4:** Consider a well behaved utility function \( u \) such that \( u^{(2p)} \), \( u^{(2p+1)} \), \( u^{(2p+2)} \) and \( u^{(2p+3)} \) exist, then:

1. \( u^{(2p)} \leq u^{(2p)} \text{ iff } [RR_{2p+1}R_{2p} - 4pR_{2p} + 2p(2p - 1)] \geq 0 \) when \( u^{(2p)} \leq 0 \)

2. \( u^{(2p+1)} \geq 0 \text{ iff } RR_{2p+1}R_{2p+2} - 2(2p + 1)RR_{2p+2} + 2p(2p + 1) \geq 0 \) when \( u^{(2p+1)} \geq 0 \)

**Proof.** (1) \( u^{(2p)}(w) = E \left( (1 + \hat{x})^2 u^{(2p)}(w(1 + \hat{x})) \right) \) is less than \( u^{(2p)}(w) \) if and only if \( x^{2p} u^{(2p)}(wx) \) is concave in \( x \). \( x^{2p} u^{(2p)}(wx) \) is concave if and only if

\[
0 \geq \left[ x^{-2p-2} u^{(2p)}(wx) \right] \left[ \left( -wx u^{(2p+1)}(u^{(2p)}) \right) \left( -wx u^{(2p+2)}(u^{(2p)}) \right) \left( -wx u^{(2p+3)}(u^{(2p)}) \right) \right] \leq 0
\]

Or equivalently if and only if

\[
x^{2p-2} (-u^{(2p)}(wx)) [R_{2p+1}R_{2p} - 4pR_{2p} + 2p(2p - 1)] \geq 0
\]

where \( RR_{2p}(w) = -w u^{(2p+1)}(w) u^{(2p)}(w) \) is the relative risk coefficient of order 2p.
Assume that $u^{(2p)}(w) \leq 0$, then the condition to be fulfilled is:

$$RR_{2p} - 2pRR_{2p} - 4pRR_{2p} + 2p(2p - 1) \geq 0$$

(2) $u^{(2p+1)}(w) = E \left\{ \int [f + x]^{2p+1} u^{(2p+1)}(w(t + x)) \right\}$ is greater than $u^{(2p+1)}(w)$ if and only if $x^{2p+1}u^{(2p+1)}(wx)$ is convex in $x$. $x^{2p+1}u^{(2p+1)}(wx)$ is convex in $x$ if and only if

$$x^{2p-1}u^{(2p+2)}(wx) \left\{ -wxu^{(2p+3)}(wx) - 2(2p + 1) \left( -wxu^{(2p+2)}(wx) + 2p(2p + 1) \right) \right\} \geq 0a$$

Or if and only if $x^{2p-1}u^{(2p+1)}(wx) \left\{ R_{2p} + R_{2p+1} - 2(2p + 1)R_{2p+1} + 2p(2p + 1) \right\} \geq 0$

Assume that $u^{(2p+1)}(w) \geq 0$, then the condition to be fulfilled is:

$$RR_{2p+1} - 2(2p + 1)RR_{2p+1} + 2p(2p + 1) \geq 0$$

The next property will allow us to sign the former two inequalities.

**Property 5:** Assume that $u$ is such that $RR_n$ is decreasing for $1 \leq n \leq N$, then:

1. $RR_{2p} - 4pRR_{2p} + 2p(2p - 1) \geq 0$ if $N = 2p$ and

2. $RR_{2p} - 2(2p + 1)RR_{2p+1} + 2p(2p + 1) \geq 0$ if $N = 2p + 1$

Or $RR_nRR_{n+1} - 2NRR_n + N(N - 1) \geq 0$

**Proof.** 1) $RR_{2p} - 4pRR_{2p} + 2p(2p - 1) = RR_{2p} \left[ RR_{2p+1} - 2p \right] - (2p) \left[ RR_{2p} - (2p - 1) \right]$

but $RR_{2p+1} \geq 2p + 1 \geq 2p$ , $RR_{2p} \geq 2p \geq 2p - 1$ and $\left[ RR_{2p+1} - (2p) \right] \geq \left[ RR_{2p} - (2p - 1) \right]$

The last inequality comes from the decrease of $RR_{2p+1}$ ($RR_{2p+2} \geq RR_{2p+1} + 1$). And the result follows.

2) $RR_{2p} - 2(2p + 1)RR_{2p+1} + 2p(2p + 1) = RR_{2p} \left[ RR_{2p+2} - (2p + 1) \right] - (2p + 1) \left[ RR_{2p+1} - 2p \right]$

but $RR_{2p+2} \geq 2p + 2 \geq 2p + 1$ , $RR_{2p+1} \geq 2p + 1 \geq 2p$ and $\left[ RR_{2p+2} - (2p + 1) \right] \geq \left[ RR_{2p+1} - 2p \right]$

The last inequality is due to the decrease of $RR_{2p+1}$. And the result follows.

$$RR_{2p} - 4pRR_{2p} + 2p(2p - 1) \geq 0 \text{ if } N = 2p \text{ and }$$

$$RR_{2p+1} - 2(2p + 1)RR_{2p+1} + 2p(2p + 1) \geq 0 \text{ if } N = 2p + 1$$

are equivalent to $RR_nRR_{n+1} - 2NRR_n + N(N - 1) \geq 0$

**Property 5 bis:** Assume that $u$ is such that $RR_n$ is decreasing. If $RR_n \geq N$ , then

$$RR_nRR_{n+1} - 2NRR_n + N(N - 1) \geq 0$$

**Proof.** $RR_nRR_{n+1} - 2NRR_n + N(N - 1) = RR_n \left[ RR_{n+1} - N \right] - N \left[ RR_n - (N - 1) \right]$

But $RR_n \geq N \geq N - 1$ , $RR_{n+1} \geq 1$ and $RR_n \geq N + 1 \geq N$. And the result follows.

For example, if $R_n^2(w) = \frac{u''(w)}{u'(w)} \geq 1$ and decreasing, then $P_n^2(w) = \frac{u''(w)}{u'(w)} \geq 2$ and $P_n^2(w)R_n^2(w) - 2R_n^2(w) \geq 0$.
We have also the following result: if $P^R_u(w) \geq 2$ and decreasing, then $T^R_u(w) = 3$ and $P^R_u(w)R^R_u(w) - 4P^R_u(w) + 2 \geq 0$.

### 4. Affiliated utility function

Following Franke, Schlesinger and Stapleton (2006), we can use the concept of affiliated function to demonstrate the properties of the direct utility function. The affiliated utility function $\tilde{u}$ is such that $\tilde{u}(\ln(w)) = u(\ln(w))$ or $\tilde{u}(\theta) = u(\theta)$ where $\theta = \ln(w)$. Dealing with multiplicative risk with the affiliated utility function is equivalent to adding $\tilde{u}$ to the wealth of the agent with utility function $u$. Therefore, multiplying the wealth of the agent with utility function $u$ by $1 + \tilde{x}$ is equivalent to adding $\tilde{e}$ to the wealth of the agent with utility function $\tilde{u}$.

The first fifth derivatives of the direct utility function and that of the affiliated utility function are related as follows:

1. $\tilde{u}'(\ln w) = u'(w)$
2. $\tilde{u}''(\ln w) = wu''(w)$
3. $\tilde{u}'''(\ln w) = wu''(w) + w^2u'''(w)$
4. $\tilde{u}^{(4)}(\ln w) = wu''(w) + 3w^2u'''(w) + w^3u^{(4)}(w)$
5. $\tilde{u}^{(5)}(\ln w) = wu''(w) + 7w^2u'''(w) + 6w^3u^{(4)}(w) + w^4u^{(5)}(w)$

We can sign the successive derivative of the affiliated utility function thanks to the coefficients of relative risk aversion, relative prudence, relative temperance and relative quintessence of the direct utility function as follows:

i) $\tilde{u}''(\ln w) \leq 0$ if and only if $u$ is risk averse ($R^R_u(w) \geq 0$).
ii) $\tilde{u}'''(\ln w) \geq 0$ if and only if $P^R_u(w) \geq 1$
iii) $\tilde{u}^{(4)}(\ln w) \leq 0$ if and only if $P^R_u(w)\llbracket T^R_u(w) - 2 \rrbracket \geq P^R_u(w) - 1$. Then, $\tilde{u}^{(4)}(\ln w) \leq 0$ if $P^R_u(w) \geq 1$ and $T^R_u(w) \geq 2$
iv) $\tilde{u}^{(5)}(\ln w) \geq 0$ if and only if $P^R_u(w)\llbracket Q^R_u(w) - 3 \rrbracket - 3\llbracket T^R_u(w) - 2 \rrbracket + P^R_u(w) - 1 \geq 0$. Thus, $\tilde{u}^{(5)}(\ln w) \geq 0$ if $P^R_u(w) \geq 1$, $T^R_u(w) \geq 3$ and $Q^R_u(w) = 1 + T^R_u(w)$ (or $T^R_u(w)$ decreases)

We have the following property because absolute risk aversion, absolute prudence, absolute temperance and absolute quintessence of the affiliated utility function are related to relative risk aversion, relative prudence, relative temperance and relative quintessence of the direct utility function.

**Property 6:** Consider a well behaved utility function $u$ such that $u'$, $u''$, $u'''$, $u^{(4)}$ and $u^{(5)}$ exist, then:

i) $R^A_u(\ln w) = R^R_u(w)$

ii) $P^A_u(\ln w) = P^R_u(w) - 1$

iii) $T^A_u(\ln w) = \frac{P^R_u(w)\llbracket T^R_u(w) - 2 \rrbracket}{P^R_u(w) - 1} - 1$

iv) $Q^A_u(\ln w) = \frac{P^R_u(w)\llbracket Q^R_u(w) - 3 \rrbracket - 3\llbracket T^R_u(w) - 2 \rrbracket + P^R_u(w) - 1}{P^R_u(w)\llbracket T^R_u(w) - 2 \rrbracket - 3P^R_u(w) + 1} - 1$

**Proof.**

i) $R^A_u(\ln w) = \frac{\tilde{u}''(\theta)}{u''(\theta)} = \frac{e^0u''(e^0)}{u'(e^0)} = \frac{wu''(w)}{u'(w)} = R^R_u(w)$
Therefore, \( v \) is relatively more risk averse than \( u \) if

\[
- \frac{\theta}{u''(0)} = - \frac{\theta}{u''(0)} \text{ is absolutely equal if and only if } \frac{1}{u''(0)} = \frac{1}{u''(0)}.
\]

Consider two utility functions \( u \) and \( v \), then:

• additive risk aversion if and only if the direct utility function is multiplicatively risk averse,
• additive risk prudence if and only if the relative prudence of the direct utility function exceeds one,
• additive risk temperance if the relative temperance of the direct utility function is greater than four or if the relative prudence of the direct utility function is greater than one and the relative temperance of the direct utility function is greater than two,
• additive risk quintessence if the relative quintessence of the direct utility function is greater than six or if the relative prudence of the direct utility function is greater than one and the relative temperance of the direct utility function is decreasing and greater than three.

This property tells us that the affiliated utility function exhibits:

\[
\begin{align*}
\text{Property 6:} & \quad \text{Consider two utility functions } u \text{ and } v, \text{ then:} \\
\text{i) } & R^A_v(w) \geq R^A_u(w) \text{ if and only if } R^A_u(lnw) \geq R^A_v(lnw) \\
\text{ii) } & R^A_v(w) \geq R^A_u(w) \text{ if and only if } P^A_u(lnw) \geq P^A_v(lnw) \\
\text{iii) } & \text{Assume that } R^A_v(w) \geq R^A_u(w), \text{ then } T^A_v(w) \geq T^A_u(w) \text{ if } T^A_u(lnw) \geq T^A_v(lnw) \\
\end{align*}
\]

\[\text{Proof. i) Property 6 says that } \frac{R^A_u(lnw)}{R^A_v(lnw)} = R^A_v(lnw). \text{ Therefore, } v \text{ is relatively more risk averse than } u \text{ if and only if } \frac{R^A_v(lnw)}{R^A_u(lnw)} \text{ is absolutely more risk averse than } \frac{\theta}{u''(0)}.\]

ii) Property 6 tells us that \( v \) is relatively more risk prudent than \( u \) if and only if \( \frac{\theta}{u''(0)} \) is absolutely more risk prudent than \( \frac{\theta}{u''(0)} \).

iii) Thanks to 6, we have:

\[
T^A_v(lnw) = \frac{P^A_u(lnw) - 1}{P^A_v(lnw)} + 1 \quad \text{or} \quad T^A_u(lnw) = 2 + \frac{\left[P^A_v(lnw) - 1\right]}{1 + T^A_u(lnw)} \\
\]

\( v \) is relatively more temperate than \( u \) if

\[
\frac{P^A_v(lnw) - 1}{P^A_v(lnw)} \geq \frac{P^A_u(lnw) - 1}{P^A_u(lnw)} + 1 + T^A_u(lnw)
\]
But the fact that \( v \) is relatively more prudent than \( u \) implies:

\[
\frac{P_v^R(w) - 1}{P_v^R(w)} \geq \frac{P_u^R(w) - 1}{P_u^R(w)}
\]

And the result follows. That is, \( v \) is relatively more temperate than \( u \) when \( v \) is relatively more prudent than \( u \) and \( \bar{v} \) is absolutely more temperate than \( \bar{u} \).

5. More prudent behavior

In this section, we provide sufficient conditions with which an agent facing a multiplicative background risk behaves in a more risk-prudent manner. That is, the coefficient of relative prudence associated with the derived utility function is greater than that of the direct utility function. We take two approaches. The first uses the notion of risk vulnerability as soon as the opposite of the marginal utility is concerned and the second a risk-neutral probability.

With the first approach, we get the following proposition which gives the conditions under which an agent behaves in a more risk-prudent manner when facing a multiplicative background risk.

**Proposition 1:** When both relative prudence and relative temperance are decreasing, an agent facing a multiplicative background risk behaves in a more risk-prudent manner compared to the case without the background risk if his/her relative prudence is greater than two.

**Proof.** Recall that dealing with multiplicative risk and a direct utility function is equivalent to dealing with additive risk and affiliated utility function. The agent will behave in a more prudent fashion when facing multiplicative risk if and only if the agent with the affiliated utility function behaves in a more prudent manner when facing additive risk. That is, \( -A \) is risk vulnerable. Gollier and Pratt (1996) give sufficient conditions for risk vulnerability, namely the coefficient of absolute risk aversion of \( -A \) is decreasing and convex or both absolute risk aversion and absolute prudence of \( -A \) are decreasing (\( -A \) is said to be standard). These two conditions become:

- absolute prudence of the affiliated utility function is decreasing and convex
- absolute prudence and absolute temperance of the affiliated utility function decrease.

Here, we consider the second condition, namely, the decrease of both absolute prudence and absolute temperance of the affiliated utility function. This turns out to be the decrease of both relative prudence and relative temperance of the direct utility function due to the following:

First, \( P_v^A(lnw) = P_v^R(w) - 1 \), thus

\[
\frac{d}{dw} P_v^R(w) = \left( \frac{dP_v^A}{dw} (lnw) \right) \frac{1}{w}
\]

And absolute prudence of the affiliated utility function and relative prudence of the direct utility function evolve in the same direction. The decrease of relative prudence of the direct utility function implies the decrease of the absolute prudence of the affiliated utility function and vice-versa.

Second, \( T_v^A(lnw) = \frac{P_v^R(w) [T_v^R(w) - 2]}{P_v^R(w) - 1} - 1 = \frac{\phi(w)}{\bar{\phi}(w)} - 1 \)

with \( T_v^A(lnw) = \frac{P_v^R(w) [T_v^R(w) - 2]}{P_v^R(w) - 1} - 1 = \frac{\phi(w)}{\bar{\phi}(w)} - 1 \) and \( \bar{\phi}(w) = [P_v^R(w) - 1] \).
We have: 

$$-\left[ \frac{d(P^R_u(w))}{dw} \right][3-T^R_u(w)] \leq -\left[ \frac{d(T^R_u(w))}{dw} \right]P^R_u(w)$$

due to $T^R_u(w) \geq 3$ (The decrease of relative prudence implies that $T^R_u(w) = 1 + P^R_u(w)$ and we know that $P^R_u(w) \geq 2$)

or equivalently 

$$-\left[ \frac{d(P^R_u(w))}{dw} \right] \leq \left[ \frac{d(P^R_u(w))}{dw} \right]T^R_u(w) - 2 - \left[ \frac{d(T^R_u(w))}{dw} \right]P^R_u(w)$$

The last inequality is $-\phi'(w) \leq -\phi'(w)$ and it tells us that $\phi'(w)$ decreases faster than $\phi'(w)$.

Therefore $\phi'(w)$ decreases and $T^R_u(w)$ also decreases.

Collecting the two conditions implies the decrease of the absolute temperance of the affiliated utility function.

Finally, both absolute prudence and absolute temperament of the affiliated utility function are decreasing. That is, the opposite of the marginal affiliated utility function is standard. And therefore risk vulnerable. In other words, the derived utility function is more prudent than the direct utility function or equivalently $\dot{u}$ is more prudent than $\dot{u}$ or $P^R_u(w) \geq P^A_u$, which is equivalent to:

$$P^R_u(w) = -\frac{\hat{u}''(w)}{\hat{u}'(w)} \geq P^R_u = -\frac{u''(w)}{u'(w)}.$$

In the second approach, like Franke, Schlesinger and Stapleton (2006) and Jokung (2004), we can express the coefficient of relative prudence of the derived utility function with a risk-neutral probability:

$$P^R_u(w) = -\frac{\hat{u}''(w)}{\hat{u}'(w)} = -\frac{E(1+x)^2u''(w+w\hat{x})}{E(1+x)^2u''(w+w\hat{x})}.$$

Let us define the risk-neutral probability $\dot{\lambda}^u$:

$$d\lambda^u_w(x) = \frac{(1+x)^2u''(w+w\hat{x})}{E(1+x)^2u''(w+w\hat{x})}.$$

Then the coefficient of relative prudence of the derived utility function is given by:

$$-\frac{\hat{u}''(w)}{\hat{u}'(w)} = E \left[ -w(1+x)u''(w+w\hat{x}) \frac{d\lambda^u_w(x)}{u''(w+w\hat{x})} \right] = E_{\lambda^u_w} \left[ -w(1+x)u''(w+w\hat{x}) \right]$$

Or equivalently:

$$P^R_u(w) = E_{\lambda^u_w} \left[ P^R_u(w(1+x)) \right].$$

The risk-neutral probability is interesting because we know the sign of the derivative of $\dot{\lambda}^u_w(x)$ with respect to $x$. The sign of $\dot{\lambda}^u_w(x)$ is the same as the sign of $2 - P^R_u(w+w\hat{x})$ because:

$$\frac{d[\dot{\lambda}^u_w(x)]}{dx} = \frac{(1+x)u''(w+w\hat{x})}{E(1+x)^2u''(w+w\hat{x})} \left[ 2 - P^R_u(w+w\hat{x}) \right].$$

If $P^R_u(w) \geq 2\lambda w$, then $\dot{\lambda}^u_w(x)$ puts more weight on lower values of $x$ than the original probability and one consequence will be $E_{\lambda^u_w}(1+x) \leq E(1+x) = 1$.

If $P^R_u(w) \leq 2\lambda w$, then $\dot{\lambda}^u_w(x)$ puts more weight on higher values of $x$ than the original probability and one consequence will be $E_{\lambda^u_w}(1+x) \geq E(1+x) = 1$. 

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We have the following proposition:

**Proposition 2:** Assume \( P_u(w) \) is convex. If one of the following conditions holds then \( \hat{u} \) is more risk-prudent than \( u \):

(i) \( P_u(w) \geq 2 \forall w \) and \( P_u(w) \) is decreasing
(ii) \( P_u(w) \leq 2 \forall w \) and \( P_u(w) \) is increasing

**Proof.** The convexity of \( P_u(w) \) implies that:

\[
P_{\hat{u}}(w) = E_{\lambda_w} \left[ P_u \left( w(1+\hat{x}) \right) \right] \geq P_u \left[ w(1+E_{\lambda_w}(\hat{x})) \right]
\]

(i) Assume that \( P_u(w) \geq 2 \), then \( w + wE_{\lambda_w}(\hat{x}) \leq w \) because \( E_{\lambda_w}(1+\hat{x}) \leq E(1+\hat{x}) = 1 \). Decreasing relative prudence leads to \( P_{\hat{u}} \left[ w + wE_{\lambda_w}(\hat{x}) \right] \geq P_u(w) \).

Therefore, \( P_{\hat{u}}(w) \geq P_u(w) \) and the result follows.

(ii) Assume that \( P_u(w) \leq 2 \), then \( w + wE_{\lambda_w}(\hat{x}) \geq w \) because \( E_{\lambda_w}(1+\hat{x}) \geq E(1+\hat{x}) = 1 \). Increasing relative prudence leads to \( P_{\hat{u}} \left[ w + wE_{\lambda_w}(\hat{x}) \right] \geq P_u(w) \).

Therefore, \( P_{\hat{u}}(w) \geq P_u(w) \) and the result follows.

We have pointed out other conditions under which an agent facing a multiplicative risk behaves in a more risk-prudent manner. These conditions are based on the convexity of the relative prudence rather than on the decrease of both relative prudence and relative temperance as in the first approach. It will be of interest to know how these conditions are related. Nevertheless, we have the following proposition if relative prudence is concave:

**Proposition 3:** Assume \( P_u(w) \) is concave. If one of the following conditions holds then \( \hat{u} \) is less risk-prudent than \( u \):

(i) \( P_u(w) \geq 2 \forall w \) and \( P_u(w) \) is increasing
(ii) \( P_u(w) \leq 2 \forall w \) and \( P_u(w) \) is decreasing

**Proof.** The proof is similar to that of proposition 1, so it is omitted.

Here we have pointed out the conditions under which an agent facing a multiplicative risk behaves in a more risk-prudent manner. The conditions are stronger than those obtained in the additive case, in which the utility function needs to be standard. This result is new because usually authors dealing with background risk focus on the risk-averse behavior.

If we consider the class of hyperbolic absolute risk averse functions (HARA), their relative temperence, their relative prudence and their relative risk aversion are related as follows:

\[
T_u^R(w) = \frac{3-\gamma}{2-\gamma} P_u(w) \quad \text{and} \quad P_u(w) = \frac{2-\gamma}{1-\gamma} R_u^R
\]

Because \( u(w) = \frac{1-\gamma}{\gamma} \left( b + \frac{aw}{1-\gamma} \right)^{1-\gamma} \) with \( b \geq 0 \) and \( a > 0 \).

Recall that:
Risk neutral utility function: \( \gamma \to 1 \)
Quadratic utility function: \( \gamma = 2 \)
Negative exponential utility function: \( \gamma \to -\infty \) and \( b = 1 \)
Power utility function: \( \gamma < 1 \) and \( b = 0 \)

If \( \gamma < 1 \), relative prudence will be convex. Therefore the former proposition reduces to the decrease of the coefficient of relative prudence if \( \gamma < 2 \). That is, decreasing relative prudence is sufficient for \( \hat{u} \) to be more risk prudent than \( u \) when \( \gamma < 2 \).
Let us consider the case of constant relative prudence utility functions. A utility function exhibits constant relative prudence if and only if:
\[-w \frac{u'''(w)}{u''(w)} = \Theta + 1.\]

Or if and only if
\[wu'''(w) + (\Theta + 1)u''(w) = 0.\]

Therefore, \[u(w) = \lambda w^{1-\Theta}\]

And the set of constant relative risk aversion (CARA) coincides with the set of constant relative prudence (CRP).

6. Preservation of decreasing relative prudence

We have an important result concerning the decrease of the coefficients of relative prudence of the direct utility function and the derived utility function. That is, if the utility function exhibits decreasing relative prudence, the derived utility function also exhibits decreasing relative prudence. Decreasing relative prudence with the direct utility function implies decreasing relative prudence with the derived utility function.

Proposition 4: If \[P_u(w)\] decreases, then \[P^R_u(w)\] decreases

Proof. We have to demonstrate that \[P_u(w) \geq 1 + P_u^R \Rightarrow P_u^R \geq 1 + P_u^R\] or equivalently that:
\[-w u^{(4)}(w) \geq 1 - w u'''(w) \Rightarrow -w E(1 + \tilde{x})^4 u^{(4)}(w + w\tilde{x}) \geq 1 - w E(1 + \tilde{x})^3 u'''(w + w\tilde{x})\]

Following Franke, Stapleton and Schlesinger (2006), we use the difference theorem (Gollier, 2001) and we must show the following implication:
\[E \left[ (1 + \tilde{x})^4 wu'''(w + w\tilde{x}) + (\lambda - 1)(1 + \tilde{x})^2 u''(w + w\tilde{x}) \right] = 0 \Rightarrow E \left[ (1 + \tilde{x})^4 wu^{(4)}(w + w\tilde{x}) + \lambda(1 + \tilde{x})^3 u''(w + w\tilde{x}) \right] \leq 0 \quad \forall \tilde{x}\]

Which holds if there is a real \( m \) such that: \( \forall \tilde{x} \)
\[1 + \tilde{x})^4 wu^{(4)}(w + w\tilde{x}) + \lambda(1 + \tilde{x})^3 u''(w + w\tilde{x}) \leq m \left[ (1 + \tilde{x})^4 wu'''(w + w\tilde{x}) + (\lambda - 1)(1 + \tilde{x})^2 u''(w + w\tilde{x}) \right] \]

Or:
\[(1 + \tilde{x})^4 u'''(w + w\tilde{x}) \left[ \frac{u^{(4)}(w + w\tilde{x})}{w u''(w + w\tilde{x}) + \lambda} \right] \leq m(1 + \tilde{x})^3 u''(w + w\tilde{x}) \left[ (1 + \tilde{x})^4 wu'''(w + w\tilde{x}) + (\lambda - 1) \right]

Or:
\[(1 + \tilde{x})^4 u'''(w + w\tilde{x}) \left[ \lambda - P_u^R \right] \leq m(1 + \tilde{x})^3 u''(w + w\tilde{x}) \left[ \lambda - 1 - P_u^R \right]\]

We can notice that because of the decrease of relative prudence, we have
\[(1 + \tilde{x})^4 u'''(w + w\tilde{x}) \left[ \lambda - P_u^R \right] \leq (1 + \tilde{x})^3 u''(w + w\tilde{x}) \left[ \lambda - 1 - P_u^R \right]\]

So, we will get our result if we can find \( m \) such that:
\[1 + \tilde{x})^4 u'''(w + w\tilde{x}) \left[ \lambda - 1 - P_u^R \right] \leq m(1 + \tilde{x})^3 u''(w + w\tilde{x}) \left[ \lambda - 1 - P_u^R \right]\]

If we take \( m = \frac{1 - \lambda}{w} \), then we have:
\[(1 + \tilde{x})^4 u'''(w + w\tilde{x}) \left[ \lambda - 1 - P_u^R \right] \leq \frac{1 - \lambda}{w}(1 + \tilde{x})^3 u''(w + w\tilde{x}) \left[ \lambda - 1 - P_u^R \right]\]
Or: 
\[-w(1 + \hat{x}) \frac{u''(w + wx)}{u''(w + wx)} \left[ \lambda - 1 - P^R_u \right] \leq -(1 - \lambda) \left[ \lambda - 1 - P^R_u \right] \]

Which is equivalent to 
\[P^R_v \left[ \lambda - 1 - P^R_u \right] \leq -(1 - \lambda) \left[ \lambda - 1 - P^R_u \right] \]

Or 
\[P^R_v \left[ \lambda - 1 - P^R_u \right] + (1 - \lambda) \left[ \lambda - 1 - P^R_u \right] \leq 0 \]

Which leads to 
\[- \left[ \lambda - 1 - P^R_u \right]^2 \leq 0 \]

And the result follows.

This result is of interest because utility functions are expected to have decreasing relative prudence. Therefore, we will have the same property with the derived utility function as soon as the direct utility function exhibits decreasing relative prudence.

7. Comparative risk prudence under multiplicative background risk

Let us consider two individuals with respective utility functions \( u \) and \( v \). Our objective is to point out conditions under which given that the one with utility function \( v \) is more risk-prudent will remain more risk-prudent in the presence of the multiplicative background risk. Or if you prefer, under which conditions \( \hat{v} \) is more risk-prudent than \( \hat{u} \) knowing that \( v \) is more risk-prudent than \( u \).

Does a multiplicative background risk preserve the risk-prudence order?

Proposition 5: Assume \( v \) is more risk-prudent than \( u \). If there is \( \chi \) such that
\[P^R_v (w) \geq \chi \geq P^R_u (w) \forall w\]
then \( \hat{v} \) is more risk-prudent than \( \hat{u} \).

Proof. If
\[P^R_v (w) \geq \chi \geq P^R_u (w) \forall w, \text{ then } P^R_v (w + wx) \geq \chi \geq P^R_u (w + wx) \forall w, x. \]
That is, \( P^R_v (w + wx) \geq \chi \) and \( \chi \geq P^R_u (w + wx) \).

By taking the expectation with respect to \( d\lambda^*_v \) and \( d\lambda^*_u \) respectively, one obtains:
\[P^R_v = E_{\lambda^*_v} \left[ P^R_v (w + wx) \right] \geq E_{\lambda^*_v} (\chi) = \chi \text{ with } d\lambda^*_v \]
and \[E_{\lambda^*_v} (\chi) = \chi \geq E_{\lambda^*_u} \left[ P^R_u (w + wx) \right] = P^R_u \text{ with } d\lambda^*_u \].

And finally,
\[P^R_v (w) \geq \chi \geq P^R_u (w) \forall w\]

And the result follows.

We have another proposition concerning the preservation of the risk-prudence order by the use of the theorem of Kihlstrom, Romer and Williams (1981):

Proposition 6: Assume \( v \) is more risk-prudent than \( u \), relative prudence of \( u \) is greater than one and either \( u \) or \( v \) exhibits decreasing relative prudence, then \( \hat{v} \) is more risk-prudent than \( \hat{u} \).

Proof. \( v \)'s being more risk-prudent than \( u \) and either \( u \) or \( v \)'s exhibition of decreasing relative prudence is equivalent to \( \tilde{v} \)'s being more risk-prudent than \( \tilde{u} \) and either \( \tilde{u} \) or \( \tilde{v} \)'s exhibition of decreasing absolute prudence because:

a) \( v \)'s being more risk-prudent than \( u \) means \( -\tilde{v} \) is more risk-averse than \( -\tilde{u} \).

b) \( R^A_v (lnw) = P^A_v (lnw) = P^R_u (w) - 1 \)
c) \( -\tilde{u} (lnw + ln(1 + x)) = -u (w(1 + x)) \)
Applying Kihlstrom, Romer and Williams' theorem leads to $-\mathcal{E} \mathcal{V}^{'} \left\{ \ln w + \ln (1 + \hat{x}) \right\}$ being more risk-averse than $-\mathcal{E} \mathcal{U}^{'} \left\{ \ln w + \ln (1 + \hat{x}) \right\}$, which is equivalent to $-\mathcal{V}^{'}$'s being more risk-averse than $-\mathcal{U}^{'}$ because the effect of the multiplicative risk on the direct utility is equivalent to the effect of the additive risk on the affiliated utility function. Finally:

$$\frac{-(\mathcal{V}^{''} w)}{-(\mathcal{V}^{'} w)} \geq \frac{-(\mathcal{U}^{''} w)}{-(\mathcal{U}^{'} w)} \text{ or } \frac{\mathcal{V}^{'''} w}{\mathcal{U}^{'''} w} \geq 2$$

And $\mathcal{V}$ is more risk-prudent than $\mathcal{U}$.

8. The optimal demand for savings with multiplicative background risk

Consider a two-period life-cycle model with an additively separable utility:

$$U(c_1, \bar{c}_2) = u(c_1) + \frac{1}{1+\delta} \mathcal{E} u(\bar{c}_2)$$

with $c_1 = y_1 - ky_1$ and $\bar{c}_2 = \left[ y_2 + ky_1(1+\bar{r}) \right](1+\hat{x})$ and where:

- $u$ is a well behaved utility function ($u^{'} \geq 0$, $u^{''} \leq 0$ and $u^{'''} \geq 0$)
- $y_1$ and $y_2$ are the present and future labor incomes
- $k$ is the rate of savings
- $r_f$ is the risk-free rate
- $\hat{x}$ is a zero-mean multiplicative background risk associated with wealth or $1+\hat{x}$ a unitary-mean multiplicative background risk
- $\delta$ is the time preference.

The agent solves:

$$\text{Max} \; u(y_1 - ky_1) + \frac{1}{1+\delta} \mathcal{E} u \left\{ \left[ y_2 + ky_1(1+\bar{r}) \right](1+\hat{x}) \right\}$$

The optimal savings rate is given by the first-order condition:

$$-y_1 \mathcal{U}^{'}(y_1 - ky_1) + y_1 \frac{1+\bar{r}}{1+\delta} \mathcal{E} \left\{ \left[ y_2 + k y_1(1+\bar{r}) \right](1+\hat{x}) \right\} = 0$$

Recall the problem to solve in case of certainty. The optimal savings rate $K_o$ satisfies:

$$\text{Max} \; u(y_1 - ky_1) + \frac{1}{1+\delta} u \left\{ y_2 + ky_1(1+\bar{r}) \right\}$$

With the following first-order condition:

$$-u'(y_1 - k_0 y_1) + \frac{1+\bar{r}}{1+\delta} u'\left[ y_2 + k_0 y_1(1+\bar{r}) \right] = 0$$

**Proposition 7:** An individual increases his or her savings in the presence of a multiplicative risk if and only if $p^{R}(w) = -w \frac{u''(w)}{u'''(w)} \geq 2$.

**Proof.** The optimal savings rate in the presence of a multiplicative background risk will be greater than that without uncertainty if and only if the first-order condition of the program with the background risk expressed at $K_o$ the savings rate without uncertainty is positive:

$$\mathcal{E} \left\{ \left[ y_2 + k_0 y_1(1+\bar{r}) \right](1+\hat{x}) \right\} \geq u'\left[ y_2 + k_0 y_1(1+\bar{r}) \right]$$

This condition is fulfilled if and only if $\Omega(w) = wu''(wx)$ is convex according to Jensen’s inequality, $\Omega(w)$ is convex if and only if $2u''(wx) + wxu'''(wx) \geq 0$ or equivalently if and only if $p^{R}(w) = -w \frac{u''(w)}{u'''(w)} \geq 2$.
This proposition generalizes the results for Leland (1968) and Sandmo (1970) who studied the case of additive risk to that of multiplicative risk. Therefore, it induces the following definition regarding the prudent behavior as do Wong and Chang (2005):

**Definition 4:** An individual is said to be multiplicatively risk-prudent if he/she increases his/her savings in the presence of a multiplicative background risk.

We can illustrate this result by considering power utility functions: \( u(w) = \frac{w^{1-\theta}}{1-\theta} \).

The coefficient of relative prudence is equal to \( \theta + 1 \).

Suppose that the multiplicative risk increases or decreases wealth by 8% with the same probability. To focus solely on the effect of the multiplicative background risk, we assume that the risk-free rate is null and that there is no time preference. We also assume that current income is 100 and final income is 50. In the event of certainty, the savings rate is 25%. Therefore, consumption is 75 today and tomorrow. The optimal savings rates are given in the following table for three utility functions:

<table>
<thead>
<tr>
<th>Utility function</th>
<th>Savings rate in the event of certainty</th>
<th>Savings rate with the background risk</th>
<th>Coefficient of relative prudence</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u(w) = w^{0.5} )</td>
<td>25%</td>
<td>24.93%</td>
<td>1.5</td>
</tr>
<tr>
<td>( u(w) = \ln(w) )</td>
<td>25%</td>
<td>25%</td>
<td>2</td>
</tr>
<tr>
<td>( u(w) = \frac{w^{-1}}{-1} = -\frac{1}{w} )</td>
<td>25%</td>
<td>25.12%</td>
<td>3</td>
</tr>
</tbody>
</table>

The coefficients of absolute prudence of the three functions are positive but only one function leads to an increase in the savings rate in the presence of the multiplicative background risk. One of them implies a decrease in the savings rate! In this last case, a prudent individual in the sense of Kimball (1990) decreases his or her savings rate when facing uncertainty with a multiplicative background risk instead of an additive background risk! This result helps to explain why an increase in the wealth risk can lead to a decrease in the savings rate.

The result from Kimball is related to additive background risk. It explains the increase in the savings rate in the presence of income risk but it fails to explain the modification of the savings rate in the presence of wealth risk or capital risk. At that time, multiplicative risk prudence is necessary.

**9. Portfolio selection with multiplicative background risk**

Consider a two-period model in which the agent receives a non random labor income \( w \) in the first period. This initial wealth will be invested in a risk-free asset \( r_f \) and in a risky asset which yields a random return \( \tilde{r} \). Let \( \alpha \) be the fraction of wealth invested in the risky asset. The agent faces a multiplicative background risk \( 1 + \tilde{x} \) such that \( E[\tilde{x}] = 0 \) and he or she chooses to maximize his or her expected utility function:

\[
\alpha \rightarrow \max \{ u(w(1+x)(1+\eta) + \alpha(\tilde{r} - \eta_f)) \}.
\]

or

\[
\alpha \rightarrow \max \{ \tilde{u}(w(1+\eta) + \alpha(\tilde{r} - \eta_f)) \}.
\]
Where \( \hat{u}(z) = Eu[ z(1+\tilde{x}) ] \)

The problem yields the following first-order condition:
\[ E \left( \tilde{r} - r \right) \hat{u}'(\tilde{w}_t) = 0 \]

where
\[ \tilde{w}_t = w \left[ (1+r_t) + \alpha(\tilde{r} - r_t) \right] \]

And the solution is
\[ \alpha = \text{ArgMax} _{\alpha} \left\{ \hat{u} \left[ w \left[ (1+r_t) + \alpha(\tilde{r} - r_t) \right] \right] \right\} \]

In the absence of the multiplicative background risk, the problem that should be solved is the following:
\[ \max \alpha \ E \left\{ u \left[ w \left[ (1+r_t) + \alpha(\tilde{r} - r_t) \right] \right] \right\} \]

The optimal proportion invested in the risky asset is given by:
\[ E \left( \tilde{r} - r \right) \hat{u}'(\tilde{w}_t) = 0 \]

And the solution is
\[ \alpha = \text{ArgMax} _{\alpha} \left\{ u \left[ w \left[ (1+r_t) + \alpha(\tilde{r} - r_t) \right] \right] \right\} \]

The difference between the two optimal proportions \( \alpha - \hat{\alpha} \) measures the effect of the presence of the multiplicative background risk on the portfolio decision. The next proposition gives the sign of this difference.

**Proposition 8:** Assume \( R_u^R \) is convex. The fraction of initial wealth optimally invested in the risky asset decreases in the presence of the multiplicative background risk if one of the two following conditions holds:
1. \( R_u^R \geq 1 \) and \( P_u^R \geq 1 + R_u^R \)
2. \( R_u^R \leq 1 \) and \( P_u^R \leq 1 + R_u^R \).

**Proof.** Franke, Schlesinger and Stapleton (2006) show that the individual with utility function \( \hat{u} \) is more risk averse than the individual with utility function \( u \) if \( R_u^R \) is convex and one of the two following conditions holds:
1. \( R_u^R \geq 1 \) and \( R_u^R \) is decreasing or
2. \( R_u^R \leq 1 \) and \( R_u^R \) is increasing.

But \( P_u^R \geq 1 + R_u^R \) if and only if \( R_u^R \) is decreasing. Therefore \( \hat{u} \) is more risk averse than \( u \). And applying Arrow (1971) leads to the fact that \( \alpha = \hat{\alpha} \).

We get the same result with the following proposition:

**Proposition 9:** When both \( R_u^R \) and \( P_u^R \) are decreasing, the fraction of initial wealth optimally invested in the risky asset decreases in presence of the multiplicative background risk.

**Proof.**
1) \( R_u^R \) is decreasing and \( P_u^R \) is also decreasing, then \( R_u^A \) and \( P_u^A \) are decreasing. That is, the affiliated utility function is standard or at least risk vulnerable. That the affiliated utility function is risk vulnerable implies that the derived utility function is more risk averse than the direct utility function. And the result follows.

We can define the feeling of standardness in the presence of multiplicative risk by replacing absolute risk aversion and absolute prudence by relative risk aversion and relative prudence. When both relative risk aversion and relative prudence will decrease, the agent will be relatively standard or relatively risk-standard. Therefore, the last proposition implies that any risk-standard agent in the multiplicative sense behaves as we expect in portfolio selection. That is, he or she reduces his or her holdings of risky assets.
10. Conclusion
This paper shows that the concept of prudence defined by Kimball (1990) corresponds to additive risk prudence, which implies that the presence of some additive background risk increases the optimal savings rate. This concept is related to the convexity of the marginal utility function. This paper introduces the concept of multiplicative risk prudence, which implies that the presence of a multiplicative background risk increases the optimal savings rate. It also introduces the notion of quintessence (the fifth element) to guarantee the decrease of the feeling of temperance.

This paper describes two conditions under which an individual facing a multiplicative risk behaves in a more risk-prudent manner. The first condition is given by the decrease of both relative prudence and relative temperance. The second assumes the convexity of the coefficient of relative prudence and either a decreasing relative prudence larger than two or an increasing relative prudence less than two.

This paper likewise presents the conditions under which the more risk-prudent individual in the absence of any multiplicative background risk remains more risk-prudent in the presence of a multiplicative background risk.

Dealing with savings demand, this paper shows that the demand for savings in the presence of some multiplicative background risk increases if and only if relative prudence is greater than two.

When the results are applied to portfolio selection and it is assumed that relative risk aversion is convex, in the presence of the multiplicative background risk, individuals demand fewer risky assets when relative prudence is greater (less) than relative aversion plus one and relative risk aversion is greater (less) than one. The result is the same when the agent is relatively standard.

11. References


Kimball M. (1990), "Precautionary Savings in the Small and in the Large", *Econometrica* 58, n°1, 53-73.


